

**Semi-analytical postbuckling analysis
of stiffened plates with a free edge**

by
Lars Brubak

**RESEARCH REPORT
IN MECHANICS**



**UNIVERSITY OF OSLO
DEPARTMENT OF MATHEMATICS
MECHANICS DIVISION**

**UNIVERSITETET I OSLO
MATEMATISK INSTITUTT
AVDELING FOR MEKANIKK**

Semi-analytical postbuckling analysis of stiffened plates with a free edge

by

Lars Brubak

Mechanics Division, Department of Mathematics,
University of Oslo

Abstract. A large deflection, semi-analytical method for pre- and postbuckling analysis of stiffened plates with a free edge is presented. The stiffeners can be oriented in both directions parallel and perpendicular to the free edge, and the stiffener spacing can be arbitrary. Ultimate strength predictions are made using a strength criterion for the plate stresses in combination with a criterion for stress limitation of the stiffener stresses. Both global and local buckling modes are captured by using a displacement field consisting of displacements representing a simply supported, stiffened plate and an unstiffened plate with a free edge. The out-of-plane and in-plane displacements are represented by trigonometric functions and linearly varying functions, defined over the entire plate. The formulations derived are implemented into a FORTRAN computer program, and numerical results are obtained for a variety of plate and stiffener geometries. Relatively high numerical accuracy is achieved with low computational efforts.

Keywords: Stiffened plates, Free edge, Postbuckling buckling analysis, Semi-analytical method.

Contents

NOTATION	v
1 INTRODUCTION	1
2 PROBLEM FORMULATION	2
2.1 Introductory comments	2
2.2 Relevant plate examples	3
2.3 Plate definition	5
3 MATERIAL LAW AND KINEMATIC RELATIONSHIPS	6
4 DISPLACEMENTS AND IMPERFECTION	7
4.1 Displacement fields	7
4.2 Imperfection shape	11
5 SOLUTION PROCEDURE	12
5.1 Incremental response propagation	12
5.2 Incremental equilibrium equations	14
5.3 Strength criteria	16
6 POTENTIAL ENERGY	18
6.1 Potential energy of the plate	18
6.1.1 Introduction	18
6.1.2 Potential bending strain energy	18
6.1.3 Potential membrane strain energy	19
6.1.4 Potential energy of an external, in-plane plate load in x -direction	19
6.1.5 Potential energy of an external lateral pressure in z -direction	20
6.2 Potential energy of stiffeners	20
6.2.1 Introduction	20
6.2.2 Potential strain energy of a stiffener parallel to the free edge	20
6.2.3 Potential strain energy of a stiffener perpendicular to the free edge	21
6.3 Potential energy of external stiffener loads for a stiffener in x -direction	22
7 FINITE ELEMENT MODEL	22

8	LOAD-DISPLACEMENT RESULTS	23
8.1	Introduction	23
8.2	Unstiffened plates with a free edge	24
8.3	Stiffened plates with a free edge and an eccentric edge stiffener	28
8.4	Stiffened plates with a free edge and three regular, eccentric stiffeners	31
9	CONCLUDING REMARKS	36
	References	36
A	Subdivision of the matrices and vectors	40
B	Rate form of energy contributions of the plate	40
B.1	\mathbf{K}^{pb} -matrices	40
B.1.1	\mathbf{K}_{ww}^{pb} -matrix	40
B.2	\mathbf{K}^{pm} -matrices	41
B.2.1	\mathbf{K}_{uu}^{pmL} -matrix	41
B.2.2	\mathbf{K}_{uw}^{pmL} -matrix	41
B.2.3	\mathbf{K}_{vv}^{pmL} -matrix	42
B.2.4	\mathbf{K}_{uw}^{pmL} -matrix	42
B.2.5	\mathbf{K}_{vw}^{pmL} -matrix	44
B.2.6	\mathbf{K}_{ww}^{pmL} -matrix	47
B.2.7	\mathbf{K}_{uw}^{pmNL} -matrix	50
B.2.8	\mathbf{K}_{uww}^{pmNL} -matrix	52
B.2.9	\mathbf{K}_{vw}^{pmNL} -matrix	53
B.2.10	\mathbf{K}_{vww}^{pmNL} -matrix	55
B.2.11	\mathbf{K}_{ww}^{pmNL} -matrix	56
B.3	The load vector $-\dot{\Lambda}\mathbf{G}$ -vector for the plate	58
B.4	\mathbf{K}^G -matrices	59
B.4.1	\mathbf{K}_{ww}^G -matrix	59
C	Rate form of energy contributions of a stiffener	59
C.1	$\mathbf{K}^{s,x}$ -matrices for stiffeners in x -direction	59
C.1.1	$\mathbf{K}_{uu}^{sL,x}$ -matrix	59
C.1.2	$\mathbf{K}_{uw}^{sL,x}$ -matrix	60
C.1.3	$\mathbf{K}_{vw}^{sL,x}$ -matrix	62
C.1.4	$\mathbf{K}_{ww}^{sNL,x}$ -matrix	64

C.1.5	$\mathbf{K}_{www}^{sNL,x}$ -matrix	66
C.1.6	$\mathbf{K}_{ww}^{sNL,x}$ -matrix	67
C.2	$\mathbf{K}^{s,y}$ -matrices for stiffeners in y -direction	69
C.2.1	$\mathbf{K}_{vv}^{sL,y}$ -matrix	69
C.2.2	$\mathbf{K}_{vw}^{sL,y}$ -matrix	70
C.2.3	$\mathbf{K}_{ww}^{sL,y}$ -matrix	71
C.2.4	$\mathbf{K}_{vw}^{sNL,y}$ -matrix	72
C.2.5	$\mathbf{K}_{vww}^{sNL,y}$ -matrix	73
C.2.6	$\mathbf{K}_{ww}^{sNL,y}$ -matrix	74

D Integrals and expressions **75**

NOTATION

Subscripts

x, y, z	Components in Cartesian coordinates
$, xy$	Differentiation with respect to x and y
f	Flange
w	Web
0	Initial

Superscripts

a	Component representing a plate with a free edge
b	Component representing a simply supported plate
c	Component representing linear variations of in-plane displacements
me	The location at the midlength of a free edge or of an edge stiffener
pb	Bending contribution of plate
pm	Membrane contribution of plate
L	Linear
NL	Nonlinear

Symbols

E	Young's modulus
ν	Poisson's ratio
E_T	Hardening modulus
f_Y	Yield strength
L	Plate length
b	Plate width
t	Plate thickness
t_w	Web thickness
h_w	Web height
h	Stiffener height
t_f	Flange thickness
b_f	Flange width

A_s	Area of the cross-section of a stiffener
I	Moment of inertia about the middle plane of the plate
u, v, w	Displacements in a Cartesian coordinate system
w_i^a, w_{ij}^b	Amplitudes for the out-of-plane displacements
u_i^a, u_{ij}^b, u^c	Amplitudes for the displacements in x -direction
v_i^a, v_{ij}^b, v^c	Amplitudes for the displacements in y -direction
w_0	Model imperfection
w_{0i}^a, w_{0ij}^b	Amplitudes of the model imperfection
N_{dof}	Total number of degrees of freedom
$\sigma_x, \sigma_y, \tau_{xy}$	In-plane stresses in a Cartesian coordinate system
$\epsilon_x, \epsilon_y, \gamma_{xy}$	In-plane strains in a Cartesian coordinate system
S_x	External stresses in the x -direction
Λ	Load factor
Π	Total potential energy
U	Strain energy
T	Potential energy of the external loads

1 INTRODUCTION

The global capacity of plated structures depends on the buckling strength of the individual stiffened plates, and each individual plate must be dimensioned to be strong enough to sustain the external loads. As alternative to the various finite element approaches and explicit design rules [1, 2], analysis using semi-analytical methods are becoming more common in structural analysis, in particular in computer based design codes [3, 4]. These methods are more accurate than the traditional explicit design formulas, and in addition more efficient than finite element analysis.

Semi-analytical methods are usually tailor-made approaches for specific cases with certain boundary conditions and load conditions, and they are not so general as finite element approaches. This will increase the computational efficiency as compared to a more general problem description, but on the other hand, restrict the range of applicability.

In the present study, plates with a free edge are of interest. For such plates, most of the semi-analytical methods available are considering the elastic buckling characteristics of unstiffened plates. Since the accuracy and convergence of the method depend on the selection of displacement fields, many researchers have studied different proposals for admissible displacement functions for such plates. A usual assumption is to use a trigonometric series in the the direction parallel to the free edge combined with polynomial functions in the perpendicular direction. In a recent work by Mittelstedt [5], various displacement functions in the direction perpendicular to the free edge were studied, including various polynomial functions and a term with a cosine function. In this work, it was found that an ordinary polynomial function was the most appropriate displacement function. The same conclusion was also drawn by Smith, Bradford and Oehlers [6], where both ordinary and orthogonal polynomials were studied. In that paper, it was found that orthogonal polynomials were computationally more expensive than simple, ordinary polynomials, despite a reduced number of terms required for adequate convergence. Ordinary polynomials have also been applied in many other works, e.g. in Madhavan and Davidson [7, 8], Qiao and Shan [9], and Yu and Schafer [10].

All the semi-analytical methods for plates with free edges mentioned above are restricted to linear elastic buckling of unstiffened plates. The main objective of this work is to develop a semi-analytical, large deflection theory model for postbuckling analysis of imperfect stiffened plates with a free edge. The model should be capable of computing the plate response and the plate stresses. The plate stresses can be used in combination with strength criteria in order to predict the ultimate strength. The proposed model is based on an incremental form of the Rayleigh-Ritz method and follows the general

solution strategy as outlined by Steen [11]. This solution strategy has also been applied by Byklum and Amdahl [12] for simply supported, regularly stiffened plates and by Brubak and Hellesland [13] for simply supported, irregularly stiffened plates. By using this solution procedure, the model is able to trace the pre- and postbuckling response, and consequently, it accounts for a possible reserve strength beyond the elastic buckling load typical for slender plates.

The stiffeners are modelled as simple beams. This implies that the model is not able to capture local failure modes of the stiffeners. Consequently, the stiffeners must be dimensioned so that premature local stiffener buckling does not occur, for instance by satisfying certain requirements according to design rules [14, 15]. The torsional stiffness of the stiffeners may be accounted for by including the torsional energy contribution. The proposed model is able to capture the interaction between local and global plate bending, including asymmetric effects typical for cases with out-of-plane bending of plates with eccentric stiffeners. By applying rotational springs, the supported edges or a part of them may be simply supported, clamped or something in between. An initial deflection is included in the model formulation in order to describe an imperfection resulting from the fabrication process of a real structure.

2 PROBLEM FORMULATION

2.1 Introductory comments

For postbuckling analysis of thin plates, a usual approach is to use the out-of-plane displacements as the only displacements that is postulated by a deflected form. Then, the in-plane stresses or displacements must be found by solving the plate compatibility equation [16]. In previously presented semi-analytical methods for simply supported plates [12, 13] this equation have been solved for by substituting an assumed Airy's stress function. For unstiffened plates with a free edge, a solution for the Airy's stress function is found by Ovesy, Loughlan and Assaee [17] using a finite strip approach. However, for semi-analytical approaches using a displacement fields defined over the entire plate, it is difficult, and maybe impossible, to find an analytical expression of Airy's stress function that satisfies both plate compatibility equation and the boundary conditions for a plate a free edge.

Another approach is to use an assumed displacement field for each displacement component u , v and w . It is this approach that is presented in this report. By introducing assumed displacements also in the in-plane directions, more degrees of freedom are needed

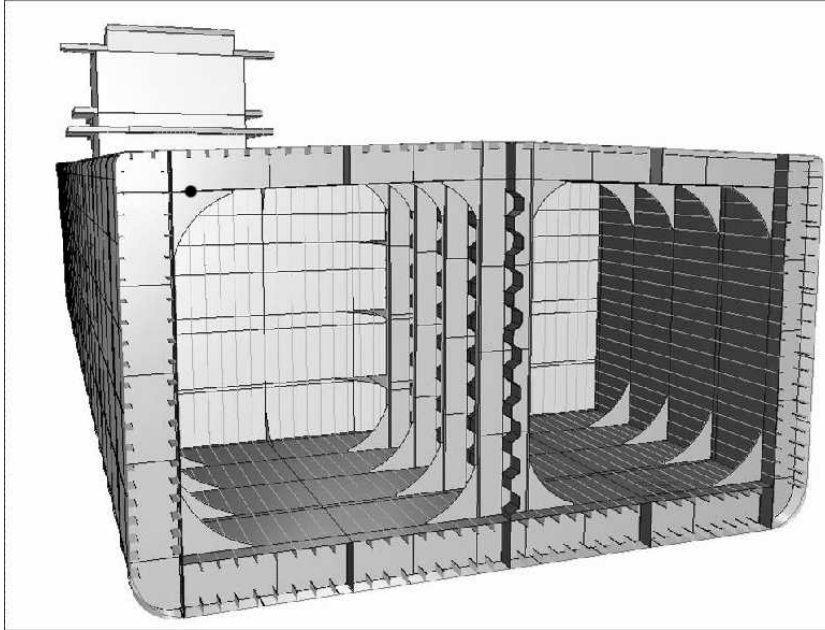


Figure 1: An illustration of a ship hull from the DNV's software program Nauticus Hull

and larger system of equations must be solved. An advantage of including in-plane displacements in the assumed displacements is that the difficulty of solving the plate compatibility equation for a stiffened plate with a plate free edge is avoided, and the stress computations becomes much more efficient with assumed in-plane displacements. Once all the displacements are known, the internal stresses and strains can be computed directly from Hooke's law.

The displacements are represented by adding together a displacement field for a simply supported, stiffened plate and a displacement field for an unstiffened plate with a free edge. As mentioned above, polynomial functions have been used in many research work for unstiffened plates with a free edge. However, if polynomials were used for stiffened plates, many terms are required for convergence, which as discussed later, may cause numerical instabilities. These numerical difficulties are avoided with the present displacement field.

2.2 Relevant plate examples

In many branches such as marine, bridge and aerospace engineering, plated structures with stiffened plates are used as the main load-carrying components. For example in Fig. 1, a typical ship hull where the structure is built up by stiffened plates is illustrated. Due

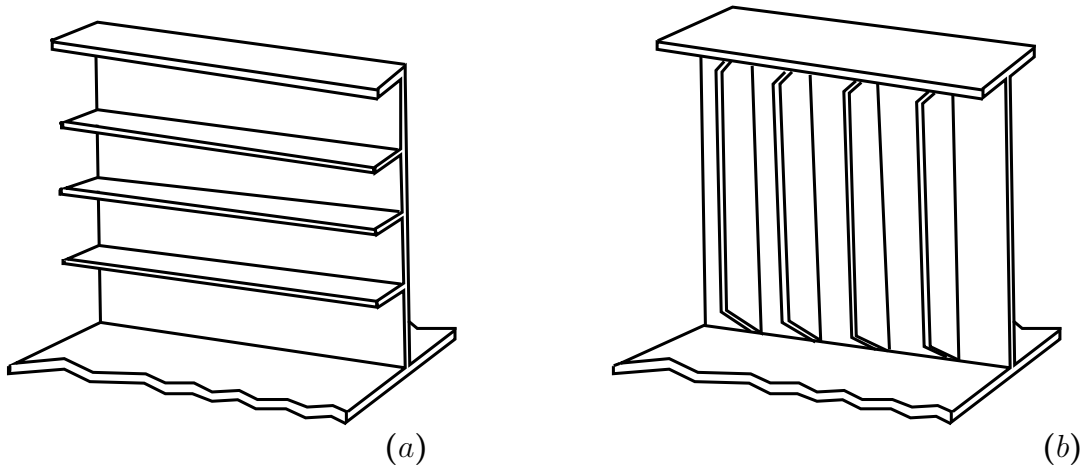


Figure 2: Typical stiffened girder examples with one free edge provided with (a) an eccentric stiffener and (b) a symmetric stiffener.

to overall bending and twisting of the ship hull, these plates are subjected to in-plane loads, and each plate must be designed to be strong enough to avoid plate collapse. A plate collapse causes material yielding and permanent displacements and this is usually not accepted. In worst case, a local plate collapse can cause an overall collapse of the entire structure.

In ship hulls, there exist both integrated plates (i.e. plates surrounded by neighbouring plates and strong girders at all edges) and plates with a free edge. Longitudinal and transverse girders, and stringer decks are examples where the plates can have a free edge. Girders and stringer decks are usually very important for the overall strength of the structure because they support the edges of the integrated plates. If these girders collapse, the entire structure may collapse.

Free edges of girders are often stiffened with either eccentric stiffeners or symmetric stiffeners as illustrated in Fig. 2(a) and (b), respectively. In addition, the interior plating of these girders can also be provided with horizontal and vertical stiffeners as shown in the figure, and the spacing between each stiffener may vary. These stiffeners can either be continuous or sniped at the edges.

Other examples where plates have a free edge are beams with channel section and with a section with flange outstands as illustrated in Fig. 3. These flange outstands can buckle locally and this may cause an overall collapse (global beam buckling) of the beam. The cases mentioned above for plates with a free edge are relevant examples where the present method is applicable.

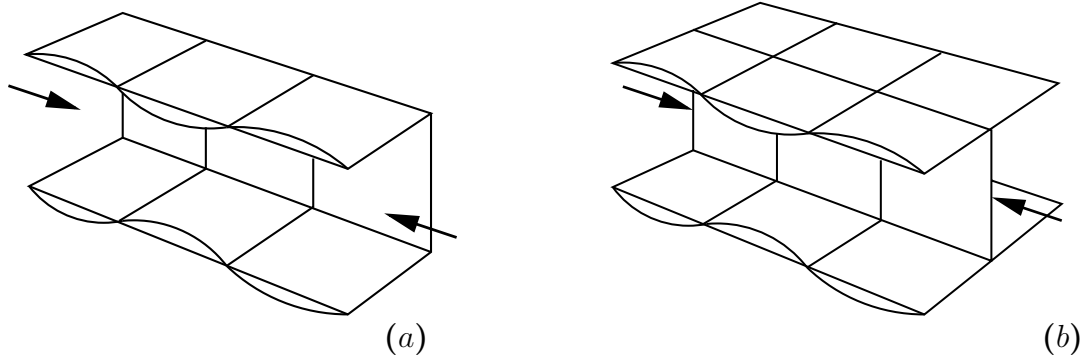


Figure 3: Local buckling of the flange outstand of a beam with (a) a channel section profile and (b) a T-section profile.

2.3 Plate definition

The rectangular plate considered can be defined with reference to Fig. 4. The plate has one free edge and it is supported in the out-of-plane direction at the three other edges. Two opposite supported edges, perpendicular to the free edge, are subjected to an external stress S_x . A plate is usually a part of a larger structure and it is assumed that the supported edges remain straight due to the neighbouring plates. In addition, the loaded edges are free to move in the in-plane directions, but they are forced to remain parallel. A supported edge boundary may be simply supported, or it may be clamped or partially restrained by adding rotational spring restraints along the edges in the same manner as described in Brubak and Hellesland [18].

The plate may be unstiffened, or it may be stiffened with one or more stiffeners oriented in the x - and y -direction. The spacing between the stiffeners can be arbitrarily chosen. The stiffeners may have different cross-section profiles, and may be eccentric, as in Fig. 4(b), or symmetric about the middle plane of the plate. The stiffeners may be end loaded (continuous stiffeners) or sniped at the ends.

The stiffeners are modelled as simple beams, and consequently, lateral deflections of the stiffeners are not accounted for. With this assumption, the stiffeners must be dimensioned such that prematurely local stiffener buckling does not occur. This can be done for instance according to existing design rules which are given to prevent local failure modes of the stiffeners. By using these design rules, the compressive stresses in the stiffeners does not exceed the critical stress for local stiffener buckling. Consequently,

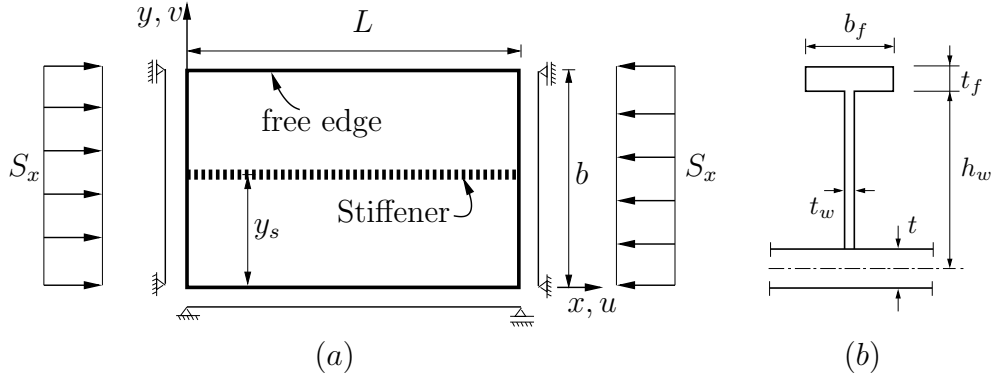


Figure 4: (a) A uniaxially loaded, stiffened plate with a free edge and three supported edges and (b) an eccentric stiffener.

in such cases, the present stiffener modelling approach neglecting local buckling of the stiffeners, seems reasonable.

The torsional stiffness of the stiffeners may be accounted for by including the torsional energy contribution. This contribution is neglected for the cases studied in the present report where only open stiffener profiles are considered, such as for instance T-profiles and flat bar profiles. This is reasonable as the torsional stiffness of stiffeners with such profiles is relatively small. In addition, it is conservative to neglect this contribution. On the other hand, for stiffener with a closed profile, the torsional stiffness may be large and it may be too conservative to neglect this stiffness contribution.

3 MATERIAL LAW AND KINEMATIC RELATIONSHIPS

For thin isotropic plates, the stresses in the thickness direction is negligibly small and it is usual to assume a plane stress condition. Further, for a material that is assumed to be linearly elastic with Young's modulus E and Poisson's ratio ν , the well known Hooke's

law is defined by

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \quad (1)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \quad (2)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = G\gamma_{xy} \quad (3)$$

where σ_x , σ_y and τ_{xy} are the in-plane stresses, and ϵ_x , ϵ_y and γ_{xy} the in-plane strains, defined positive in tension. The total strain at a distance z from the middle plane of the plate can be written as

$$\epsilon_x = \epsilon_x^{pm} - zw_{,xx} \ , \quad \epsilon_y = \epsilon_y^{pm} - zw_{,yy} \ , \quad \gamma_{xy} = \gamma_{xy}^{pm} - 2zw_{,xy} \quad (4)$$

where the first terms represent the membrane strains and the second terms expressed by out-of-plane displacements w are the bending strains. These out-of-plane displacements w are additional to an initial imperfection. The conventional notation $w_{,xy}$ for $\partial^2 w / \partial x \partial y$, etc., is adopted. The bending strain distribution complies with Kirchhoff's assumption [19] that normals to the middle plane remain normal to the deflected middle plane. For the membrane strains, the classical large deflection theory [20] is used (large rotations, but small in-plane strains). The in-plane membrane strains are defined by

$$\epsilon_x^{pm} = u_{,x} + \frac{1}{2}w_{,x}^2 + w_{0,x}w_{,x} \quad (5)$$

$$\epsilon_y^{pm} = v_{,y} + \frac{1}{2}w_{,y}^2 + w_{0,y}w_{,y} \quad (6)$$

$$\gamma_{xy}^{pm} = u_{,y} + v_{,x} + w_{,x}w_{,y} + w_{0,x}w_{,y} + w_{0,y}w_{,x} \quad (7)$$

for a plate with an initial out-of-plane imperfection w_0 . Here, u and v are the displacements of the middle plane of the plate in the x - and y -direction, respectively. These strains were given by Marguerre [16] in order to include an initial imperfection in the von Karman's plate theory [19].

4 DISPLACEMENTS AND IMPERFECTION

4.1 Displacement fields

The displacement field in each direction consists of a field representing an unstiffened plate with a free edge, identified by a super index 'a', and a simply supported, stiffened

plate, identified by a super index 'b'. In addition, a linear in-plane displacement field, identified by a super index 'c', is added to the displacement field in the x - and y -direction in order to account for linear variations due to the in-plane displacements of the edges. The displacement fields are given by

$$w = w^a + w^b \quad (8)$$

$$u = u^a + u^b + u^c \quad (9)$$

$$v = v^a + v^b + v^c \quad (10)$$

where

$$w^a(x, y) = \sum_{i=1}^{M_{wa}} w_i^a \frac{y}{b} \sin\left(\frac{\pi i x}{L}\right) \quad (11)$$

$$w^b(x, y) = \sum_{i=1}^{M_{wb}} \sum_{j=1}^{N_{wb}} w_{ij}^b \sin\left(\frac{\pi i x}{L}\right) \sin\left(\frac{\pi j y}{b}\right) \quad (12)$$

$$u^a(x, y) = \sum_{i=1}^{M_{ua}} u_i^a \frac{y}{b} \sin\left(\frac{\pi i x}{L}\right) \quad (13)$$

$$u^b(x, y) = \sum_{i=1}^{M_{ub}} \sum_{j=1}^{N_{ub}} u_{ij}^b \sin\left(\frac{\pi i x}{L}\right) \sin\left(\frac{\pi j y}{b}\right) \quad (14)$$

$$u^c(x, y) = u^c \frac{x}{L} \quad (15)$$

$$v^a(x, y) = \sum_{i=1}^{M_{va}} v_i^a \frac{y}{b} \cos\left(\frac{\pi i x}{L}\right) \quad (16)$$

$$v^b(x, y) = \sum_{i=1}^{M_{vb}} \sum_{j=1}^{N_{vb}} v_{ij}^b \sin\left(\frac{\pi i x}{L}\right) \sin\left(\frac{\pi j y}{b}\right) \quad (17)$$

$$v^c(x, y) = v^c \frac{y}{b} \quad (18)$$

where w_i^a , w_{ij}^b , u_i^a , u_{ij}^b , u^c , v_i^a , v_{ij}^b , v^c are amplitudes, L the plate length and b the plate width. In these displacement fields, the total number of degrees of freedom is

$$N_{\text{dof}} = M_{ua} + (M_{ub} \times N_{ub}) + M_{va} + (M_{vb} \times N_{vb}) + M_{wa} + (M_{wb} \times N_{wb}) + 2 \quad (19)$$

For the displacement fields representing a simply supported plate (Eqs. 12, 14 and 17), similar fields with only one term in each direction are used in Bazant [20] to study

simply supported, unstiffened plates. By including more terms in the displacement fields in each direction, it is also possible to model stiffened plates in the same manner as in Brubak et al. [13, 18, 21]. For the displacement fields representing a plate with a free edge (Eqs. 11, 13 and 16), each displacement component consists of a trigonometric series in the x -direction in the same manner as in Ovesy, Loughlan and GhannadPour [22], and a linear variation in y -direction.

Both the displacement fields representing a simply supported plate (with super index 'a') and an unstiffened plate with a free edge (with super index 'b') account for the in-plane and out-of-plane variations due to out-of-plane bending. In these fields, the in-plane displacements of the loaded edges in the direction of the load is not included. Consequently, it is necessary to include the displacements fields with super index 'c' (Eqs. 15 and 18) to account for in-plane edge displacements. For instance, if an unstiffened plate without initial imperfections is analysed by the present model, no out-of-plane displacements occur and the only non-zero displacement amplitudes are u^c and v^c . The latter component v^c accounts for the extension of the plate in y -direction due to Poisson's ratio.

Boundary conditions

The displacement fields represent a plate with three simply supported edges and one free edge. The displacements at the supported boundaries can be related to the coordinate system in Fig. 4 and expressed by

$$u(0, y) = 0, \quad u(L, y) = u^c, \quad v(0, y) = 0 \quad (20)$$

$$w(0, y) = 0, \quad w(L, y) = 0, \quad w(x, 0) = 0 \quad (21)$$

Although the plate has three simply supported edges, it is also possible to model rotationally restraint edges by using rotational springs. In the out-of-plane displacements fields, each term in the series of sine functions represents a simply supported condition. However, by adding together the terms, the sine series are, in combination with rotational springs along the supports, nearly able also to describe fully or partially restrained conditions. If plate edges, or portions of edges, are partly or fully clamped, additional strain energy contributions have to be added.

For clamped plates, it may be more convenient to assume an out-of-plane displacement with a series of cosine functions included. To achieve the same accuracy, a higher number of degrees of freedom (number of terms) will normally be required with a sine field than with a field that satisfies the kinematic boundary conditions more appropriately such as

cosine functions. Rotational edge restraints are discussed in more detail by Brubak et al. [18, 23], and by Byklum and Amdahl [12].

Discussion/comments

Some comments to the chosen displacement fields might be in order. In the assumed displacement field representing a plate with free edge, variations in the y -direction are accounted for by including linear polynomial functions. As mentioned before, polynomial functions in the y -direction have been used for analysing plates with a free edge in many research works, e.g. Mittelstedt [5], Madhavan and Davidson [7, 8], Qiao and Shan [9], and Yu and Schafer [10]. In these works, unstiffened plates were studied, and for such plates it is not necessary to use many terms in order to achieve satisfactory results.

In preliminary stages of the present work, displacements with polynomial functions with many terms were studied. In principle, the more terms that are included in the polynomial function, the more exact the solution becomes. However, numerical tests using a polynomial function showed that numerical problems occur if many polynomial terms are included. In these tests, an eigenvalue problem was established for the assumed displacement field

$$w^{po}(x, y) = \sum_{i=1}^{M_w} \sum_{j=1}^{N_w} w_{ij}^{po} \left(\frac{y}{b}\right)^j \sin\left(\frac{\pi i x}{L}\right) \quad (22)$$

where w_{ij}^{po} denotes the displacement amplitudes. Prior to these numerical tests, it was intended to use such polynomial functions for the out-of-plane displacements instead of the combined displacement field defined by Eq. 11 and 12. In order to describe the displacements for an unstiffened plate, a polynomial with 3 or 4 terms in Eq. 22 will probably be enough. For such few terms, no numerical problems occurred in the test. However, by using Eq. 22 for a stiffened plate, many terms must be included to describe the displacements, and numerical problems occur.

For the assumed displacement fields used in the present method (Eqs. 8-18), these numerical problems did not occur. The approximation of using only a linear variation in the y -direction is partly compensated for by adding the trigonometric series representing a simply supported stiffened plate (u^b, v^b, w^b). This will be demonstrated later.

4.2 Imperfection shape

The initial imperfection shape is represented using the same field as for the additional out-of-plane displacements. This imperfection can be written as

$$w_0 = w_0^a + w_0^b \quad (23)$$

where

$$w_0^a(x, y) = \sum_{i=1}^{M_{wa}} w_{0i}^a \frac{y}{b} \sin\left(\frac{\pi i x}{L}\right) \quad (24)$$

$$w_0^b(x, y) = \sum_{i=1}^{M_{wb}} \sum_{j=1}^{N_{wb}} w_{0ij}^b \sin\left(\frac{\pi i x}{L}\right) \sin\left(\frac{\pi j y}{b}\right) \quad (25)$$

Here, w_{0i}^a and w_{0ij}^b are displacement amplitudes. These amplitudes must be found in order to describe the imperfection.

In the results presented later, the first eigenmode is used for the imperfection shape. This is a usual approach in computational methods, and this eigenmode is computed in the same manner as described in detail in Brubak et al. [18]. The resulting eigenvalue problem in the common, bold face notation can be written as

$$(\mathbf{K}_e^M + \Lambda^e \mathbf{K}_e^G) \mathbf{w}^e = \mathbf{0} \quad (26)$$

where Λ^e is the eigenvalue and \mathbf{w}^e the corresponding eigenmode. \mathbf{K}_e^M and \mathbf{K}_e^G are the material and geometrical stiffness matrix, respectively. In this eigenvalue problem, the amplitudes for the out-of-plane displacements are the only unknowns. The interaction between the out-of-plane displacements and the in-plane displacements due to eccentric stiffeners is accounted for in a simplified manner. An eccentric stiffener is included in the same manner as a symmetric stiffener, but with an effective second moment of area I_e computed with an effective plate width b_e . In the same manner as in Brubak et al. [18], an effective plate width $b_e = 30t$ is used for a stiffener located at the interior plating, where t is the plate thickness. For an edge stiffener, the effective plate width is reduced to $b_e = 15t$ as supported by Eiding [24]. Similar reduction for the effective plate width for edge stiffeners was used by Rhodes [25] in another context.

In order to obtain a more correct interaction between the stiffener and the plate in an eigenvalue procedure, a method such as presented by Bedair [26] could have been used. In that method, both out-of-plane and in-plane displacements are introduced as unknowns. The approximation of using an effective second moment of area, in the present procedure

for obtaining the imperfection shape, is considered more than adequate. This has also been verified by Brubak and Hellesland [18, 21]

In principle, any shape of imperfection can be used in the present model. Another common approach for stiffened plates is to use an imperfection shape in which a global and a local imperfection mode is added together [27, 28]. The global mode may be taken equal to the displacement field of a plate without stiffeners. The local mode can be found by performing a linear elastic buckling analysis where the out-of-plane displacements along the stiffeners are prevented by using strong translational springs. Alternatively, a measured imperfection in a real plate may approximately be represented by the assumed displacement field. Any imperfection shape can be modelled by using enough terms in the displacement field.

5 SOLUTION PROCEDURE

5.1 Incremental response propagation

The postbuckling response is traced using an incremental procedure presented by Steen [11] in which an arc length parameter is used as a propagation (incrementation) parameter. By using an arc length parameter, this procedure is more general than methods with pure load or pure displacement control, and a complex plate response can be handled, including snap-through and snap-back equilibrium curves. This procedure have been applied in several other research works [12, 13, 29, 30] in which the out-of-plane displacements were the only assumed displacements. Unlike in those studies, also the in-plane displacements are included in the assumed displacement fields in the present work. This complicates the expressions in the incremental response propagation and coupling terms between the in-plane and out-of-plane displacements appear in the equations that describe the plate response.

In the large deflection theory, the equilibrium equations obtained using the Rayleigh-Ritz method are nonlinear in the displacements. In order to avoid solving nonlinear equations directly, the equilibrium equations are solved incrementally by computing the rate form of the equilibrium equations with respect to an arc length parameter η . The change in the arc length parameter can be related directly to a change in the external stresses and displacements. For an external applied stress that is changing proportionally with a load parameter Λ , this relation is illustrated graphically in Fig. 5. In the limit, as

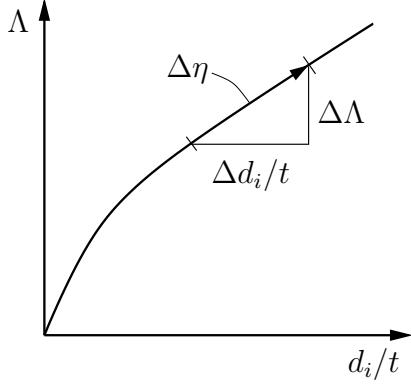


Figure 5: Illustration of the relationship between $\Delta\eta$, a load increment $\Delta\Lambda$ and an increment in the displacements for a case with one amplitude d_i .

the increment size approaches zero, it can be expressed as

$$\dot{\Lambda}^2 + \sum_{i=1}^{N_{\text{dof}}} \frac{d_i^2}{t^2} = 1 \quad (27)$$

where N_{dof} is to total number of degrees of freedom and d_i is a vector consisting of a chosen assembly of all the displacement amplitudes. This vector can be written as

$$d_i = \left[u_1^a, \dots, u_{M_{ua}}^a, u_{11}^b, u_{12}^b, \dots, u_{M_{ub}N_{ub}}^b, v_1^c, \dots, v_{M_{va}}^a, v_{11}^b, \right. \\ \left. v_{12}^b, \dots, v_{M_{vb}N_{vb}}^b, v_1^c, w_1^a, \dots, w_{M_{va}}^a, w_{11}^b, w_{12}^b, \dots, w_{M_{wb}N_{wb}}^b \right] \quad (28)$$

In Eq. 27, the plate thickness t is introduced in order to obtain dimensional consistency. A dot above a symbol ($\dot{\Lambda}$, etc.) means differentiation with respect to the arc length parameter η , which can be considered a pseudo-time.

In the incremental procedure, the load parameter Λ and displacement amplitudes d_i are functions of the arc length parameter η . For an increment $\Delta\eta$ along the equilibrium curve from point “ k ” to “ $k + 1$ ”, a Taylor series expansion gives

$$d_i^{k+1} = d_i^k + \dot{d}_i^k \Delta\eta + \frac{1}{2} \ddot{d}_i^k \Delta\eta^2 + \dots \quad (29)$$

$$\Lambda^{k+1} = \Lambda^k + \dot{\Lambda}^k \Delta\eta + \frac{1}{2} \ddot{\Lambda}^k \Delta\eta^2 + \dots \quad (30)$$

The second and higher order terms are neglected in the present work, resulting in a first order expansion. The approximation including only the first order expansion is usually referred to as the Euler or Euler-Cauchy method. In other works, such as in Steen [11]

and Byklum [27], it is shown how to include the second order terms. However, in the latter work, it was found that significant computational gains (efficiency) are not achieved compared to the Euler method with smaller increments.

The accuracy of the present method can also be improved by using equilibrium corrections after each increment, for instance as in Riks' arc length method [31], or alternatively by using an improved Euler method (Heun's method), which is a predictor-corrector method [32]. However, these improvements are also computationally costly and will not likely result in significant computational gains although larger increments can be used.

5.2 Incremental equilibrium equations

Equilibrium is satisfied using the principle of stationary potential energy (Rayleigh-Ritz method) on an incremental form (rate form) as mentioned above. The incremental form of the stationary potential energy principle, $\delta\dot{\Pi} = 0$, where Π is the total potential energy, leads to N_{dof} linear equations in $N_{\text{dof}} + 1$ unknowns. The additional equation required is given by Eq. 27. The incremental form of the stationary potential energy principle leads to

$$\frac{\partial\dot{\Pi}}{\partial d_i} = \frac{\partial}{\partial\eta} \frac{\partial\Pi}{\partial d_i} = K_{ij}\dot{d}_j + G_i\dot{\Lambda} = 0 \quad (31)$$

where

$$K_{ij} = \frac{\partial^2\Pi}{\partial d_i\partial d_j} \text{ and } G_i = \frac{\partial^2\Pi}{\partial d_i\partial\Lambda} \quad (32)$$

Here, K_{ij} is a generalised, incremental (tangential) stiffness matrix, $-G_i\dot{\Lambda}$ is a generalised, incremental load vector. Above, the index notation with the Einstein summation rule for repeated indexes is adopted. Alternatively, in the common, bold face matrix notation, the final set of equations, including Eq. 27, can be given by

$$\mathbf{K}\dot{\mathbf{d}} + \mathbf{G}\dot{\Lambda} = 0 \quad (33)$$

$$\dot{\Lambda}^2 + \frac{1}{t^2}\dot{\mathbf{d}}^T\dot{\mathbf{d}} = 1 \quad (34)$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{uv} & \mathbf{K}_{uw} \\ \mathbf{K}_{vu} & \mathbf{K}_{vv} & \mathbf{K}_{vw} \\ \mathbf{K}_{wu} & \mathbf{K}_{wv} & \mathbf{K}_{ww} \end{bmatrix} \quad (35)$$

$$\mathbf{d} = [\mathbf{u}, \mathbf{v}, \mathbf{w}]^T \quad (36)$$

$$\mathbf{G} = [\mathbf{G}_u, \mathbf{G}_v, \mathbf{G}_w]^T \quad (37)$$

In these expressions, the matrix \mathbf{K} is divided into submatrices and the vector \mathbf{G} into subvectors, e.g., \mathbf{K}_{uu} , \mathbf{K}_{uv} , \mathbf{G}_u , etc. Further, these submatrices and subvectors can be subdivided for each displacement amplitude with super index 'a', 'b' and 'c', e.g., $\mathbf{G}_u = [\mathbf{G}_{ua}, \mathbf{G}_{ub}, \mathbf{G}_{uc}]^T$, $\mathbf{u} = [\mathbf{u}_a, \mathbf{v}_b, \mathbf{u}_c]^T$, etc. The vector \mathbf{d} an assembly of the displacement amplitudes and in index notation this vector is given in Eq. 28. More details on the subdivision of the matrices and vectors are given in Appendix A.

The incremental stiffness matrix and incremental load vector consist of contributions both from the plate and the stiffener, which can be expressed as

$$\mathbf{K} = \mathbf{K}^{pb} + \mathbf{K}^{pm} + \mathbf{K}^s \quad (38)$$

$$\mathbf{G} = \mathbf{G}^p + \mathbf{G}^s \quad (39)$$

where \mathbf{K}^{pb} and \mathbf{K}^{pm} are the bending stiffness matrix and the membrane stiffness matrix of the plate, respectively, and \mathbf{K}^s is the stiffness matrix of the stiffeners. In Eq. 39, the load vector is separated into a plate contribution \mathbf{G}^p and a stiffener contribution \mathbf{G}^s . The latter contribution is zero if the stiffeners are not end loaded, which is the case for sniped stiffeners.

The bending stiffness matrix of the plate \mathbf{K}^{pb} is independent of the displacement amplitudes and is a constant contribution of the stiffness matrix. On the other hand, both the membrane stiffness matrix of the plate \mathbf{K}^{pm} and the stiffness matrix of the stiffener \mathbf{K}^s is dependent on the displacement amplitudes. These two matrices can be divided into linear (L) and a nonlinear contribution (NL), and they can be written as

$$\mathbf{K}^{pm} = \mathbf{K}^{pmL} + \mathbf{K}^{pmNL} \quad (40)$$

$$\mathbf{K}^s = \mathbf{K}^{sL} + \mathbf{K}^{sNL} \quad (41)$$

The linear contributions are constant, and the nonlinear contributions are dependent of the displacement amplitudes. The analytical expressions of stiffness matrices and the load vectors for the plate are given in Appendix B. For the stiffeners, the stiffness matrix is given in Appendix C, and it is divided into contributions, $\mathbf{K}^{sL,x}$ and $\mathbf{K}^{sNL,x}$, for the stiffeners in the x -direction, and contributions, $\mathbf{K}^{sL,y}$ and $\mathbf{K}^{sNL,y}$, for stiffeners in the y -direction.

In the propagation process, the displacement rates $\dot{\mathbf{d}}$ and the load parameter rate $\dot{\Lambda}$ can be determined from Eqs. 33 and 34 at a specific state "k". The solution at state "k + 1" is then obtained from linear Taylor series expansion as

$$\mathbf{d}^{k+1} = \mathbf{d}^k + \dot{\mathbf{d}}^k \Delta\eta; \quad \Lambda^{k+1} = \Lambda^k + \dot{\Lambda}^k \Delta\eta \quad (42)$$

By using this first order expansion, the solution propagation continues until a given criterion is reached. Such criterion can for instance be a certain level of the external stress or a specified strength criterion. Various strength criteria are discussed more in detail in section below.

5.3 Strength criteria

Strength criteria can be used in combination with the incremental postbuckling procedure outlined above in order to predict the ultimate strength of stiffened plates with a free edge. Typically, such strength criteria are applied in critical parts of the plate. For a plate with predominantly in-plane loads, the critical parts are the parts of the plate with the largest stiffness, which are at the supported edges and, in addition, at the stiffeners in local bending cases. Due to increasing out-of-plane displacements, membrane stresses are redistributed from the interior of the plate to these parts. Finally, when the capacity of the critical parts are exhausted, no more stresses can be redistributed, and additional in-plane loading cannot be applied without causing collapse.

In Brubak and Hellesland [33], various strength criteria for the plate stress were studied, including a “no yield criterion”, a “membrane stress criterion” and an “interaction curve criterion”. The first criterion was applied in the entire plate, and the two latter criteria were applied in critical parts of the plate. For completeness and convenience of the reader, the main aspects and conclusions from that paper are given below.

In the “no yield criterion”, the analysis is terminated if the total stresses, including bending and membrane stresses, cause yielding in the any parts of plate. In combination with a similar stress limitation for the stiffeners, this criterion is normally too conservative in the context of ultimate strength predictions, but since yielding will give permanent deformations, this may be a sound design criterion in practise.

The “membrane stress criterion”, defined by first yield at the midplane ($z = 0$) in critical parts, proved to be somewhat non-conservative in cases where the bending stresses at critical parts of the plate are important. It was found that the bending stresses are important in large deflection analysis of relatively thick plates with an irregular stiffener arrangement.

The “interaction curve criterion” reflects a stronger effect of the bending stresses. This criterion was expressed as

$$\left(\frac{\sigma_e^{pm}}{f_Y}\right)^2 + \frac{1}{\alpha} \frac{\sigma_{e,\max}^{pb}}{f_Y} = 1 \text{ where } \alpha = 1.5 \quad (43)$$

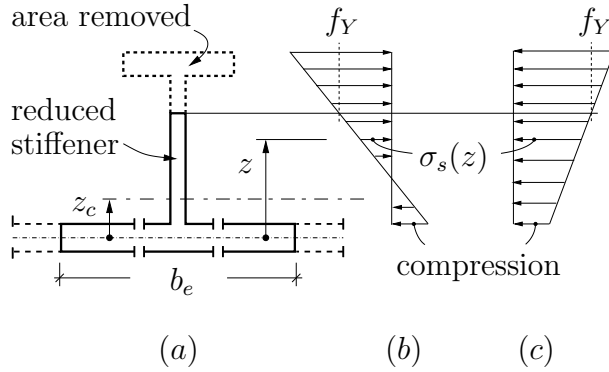


Figure 6: (a) Yielded area removed for the cross-section with the maximum elastic stress, and stress distribution due to global bending in (b) positive z -direction and (c) negative z -direction.

where f_Y is the yield strength, σ_e^{pm} is the equivalent membrane stress and $\sigma_{e,\max}^{pb}$ is the equivalent bending stress in the outer plate fibres ($z = t/2$). For regularly stiffened plates subjected to in-plane loads, the bending stresses at the critical parts are small, and for these cases it was found that predictions with this criterion gave the same ultimate strength as predictions using the “membrane stress criterion”.

For global bending cases, the stresses in the stiffeners may be important for the ultimate strength, and for such cases a criterion for stress limitation in the stiffener stresses is required. In Brubak and Hellesland [33], two different stiffener criteria were presented. These were “no stiffener yield criterion” and a “stiffener reduction criterion”. In the former criterion, the analysis is terminated when the stress at the outer fibre of the stiffener reaches the yield stress. As mentioned above, this criterion in combination with a similar for the plate may be too conservative in ultimate strength predictions. In the latter criterion, the progression of plasticity in the stiffener is accounted for in a simplified manner by removing the yielded parts of the stiffeners as illustrated in Fig. 6. For a stiffener, the cross sections along the entire stiffener are reduced with the yielded area in the most strained cross section of the stiffener.

It was found [33] that the “interaction curve criterion” for the plate in combination with the “stiffener reduction criterion” generally gave the most accurate results in the context of ultimate strength prediction. Strength prediction results by the present semi-analytical method using a combination of these two criteria will be presented later.

6 POTENTIAL ENERGY

6.1 Potential energy of the plate

6.1.1 Introduction

The potential energy of the plate consists of a strain energy contribution and an energy contribution due to external loads. The potential strain energy of the plate gives contribution to the incremental stiffness matrix, and the potential energy of the external stresses along the plate edges contribute to the incremental load vector. Each contribution of the potential energy of the plate is given below, and due to the large and complex expressions, the rate form of these contributions are given separately in Appendix *B*.

For thin plates, the potential strain energy U^p can be given by

$$U^p = \frac{1}{2} \int_V \boldsymbol{\sigma}^T \boldsymbol{\epsilon} dV \quad (44)$$

where $\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \tau_{xy}]^T$, $\boldsymbol{\epsilon} = [\epsilon_x, \epsilon_y, \gamma_{xy}]^T$ and V is the volume of the plate. It is usual to divide the strain energy into a part due to membrane stretching of the middle plane of the plate and a part due to bending about the middle plane of the plate. This can be written as

$$\begin{aligned} U^p &= \frac{1}{2} \int_V (\boldsymbol{\sigma}^{pm} + \boldsymbol{\sigma}^{pb})^T (\boldsymbol{\epsilon}^{pm} + \boldsymbol{\epsilon}^{pb}) dV \\ &= \frac{1}{2} \int_V (\boldsymbol{\sigma}^{pm})^T \boldsymbol{\epsilon}^{pm} dV + \frac{1}{2} \int_V (\boldsymbol{\sigma}^{pb})^T \boldsymbol{\epsilon}^{pb} dV \\ &= U^{pm} + U^{pb} \end{aligned} \quad (45)$$

where U^{pm} and U^{pb} are the potential membrane and bending strain energy, respectively. Since the bending stress is linearly varying over the plate thickness and is acting about the middle plane of the plate, the coupling terms between the membrane and bending contributions disappear when integrating over the plate thickness.

6.1.2 Potential bending strain energy

By substituting Hooke's law into bending part of Eq. 45 and then integrating this contribution over the plate thickness, the elastic strain energy contribution from bending of the plate can be written as [19]

$$U^{pb} = \frac{D}{2} \int_0^b \int_0^L \left((w_{,xx} + w_{,yy})^2 - 2(1 - \nu)(w_{,xx}w_{,yy} - w_{,xy}^2) \right) dx dy \quad (46)$$

where $D = Et^3/12(1 - \nu^2)$ is the plate bending stiffness and t is the plate thickness. By substituting the assumed displacement field, an analytical solution of this integral may be derived. This energy contribution is of quadratic order in the displacement amplitudes. Thereby, it gives a constant contribution to the incremental plate stiffness matrix since this matrix is obtained by differentiation twice with respect to the displacement amplitudes (Eq. 32). Consequently, it is necessary to computed this matrix only once. The bending stiffness matrix of the plate on rate form is given in Appendix B.

6.1.3 Potential membrane strain energy

By substituting Hooke's law into membrane part of Eq. 45 the elastic membrane strain energy of the plate can be written as [19]

$$U^{pm} = \frac{C}{2} \int_0^b \int_0^L \left((\epsilon_x^{pm})^2 + (\epsilon_y^{pm})^2 - 2\nu(\epsilon_x^{pm})(\epsilon_y^{pm}) - \frac{1-\nu}{2}(\gamma_{xy}^{pm})^2 \right) dx dy \quad (47)$$

where $C = Et/(1 - \nu^2)$ is the extensional stiffness of the plate. By substituting the membrane strains from Eqs. 5-7 and the assumed displacement fields into this equation, an analytical solution of this integral may be derived. The resulting expression can be separated into a term U^{pmL} that is quadratic in the displacement amplitudes and a term U^{pmNL} that is of a higher order in the amplitudes. The membrane strain energy can then be written as

$$U^{pm} = U^{pmL} + U^{pmNL} \quad (48)$$

The first term in Eq. 48 gives constant contribution \mathbf{K}^{pmL} to the total incremental plate stiffness matrix in Eq. 32. Thus, this matrix must be calculated only once, and does not affect the computation time significantly. The second term in Eq. 48 is contributes to the nonlinear, incremental stiffness matrix \mathbf{K}^{pmNL} . This matrix is dependent of the displacement amplitudes and consequently, it must be calculated for every increment in the solution propagation described in Section 5. Thus, this matrix affect the computational efficiency significantly.

6.1.4 Potential energy of an external, in-plane plate load in x -direction

The potential energy of an external, in-plane load acting on the plate in the x -direction is given by

$$T^{p,x} = -\Lambda S_{x0} t b \Delta u \quad (49)$$

where S_{x0} are a reference stress and $\Delta u = u^c$ is the plate shortening in the x -direction. Eq. 49 gives a contribution to the incremental load vector $-\mathbf{G}\dot{\Lambda}$. An analytical expression for this vector is given in Appendix B.

6.1.5 Potential energy of an external lateral pressure in z -direction

The potential energy of an external lateral pressure acting on the plate in the z -direction can be given by

$$T^{p,z} = - \int_0^b \int_0^L p w dx dy \quad (50)$$

where $p = p(x, y)$ is the lateral pressure. This contribution gives a constant contribution to the incremental load vector. More details on how to include an external, lateral pressure can be found in Byklum [27]. This load case is not studied in detail in the present report.

6.2 Potential energy of stiffeners

6.2.1 Introduction

The potential energy of the stiffener consists of a strain energy contribution and an energy contribution due to external stiffener loads. The potential strain energy of the stiffener gives contribution to the incremental stiffness matrix, and the rate form of this energy contribution is given separately in Appendix C. The potential energy of external stiffener loads for a stiffener in x -direction is also given below. In the present report, end loaded stiffeners are not considered, and the rate form of this energy contribution is not given.

6.2.2 Potential strain energy of a stiffener parallel to the free edge

The elastic strain energy of a stiffener parallel to free edge is given by

$$\begin{aligned} U^{s,x} &= \frac{E}{2} \int_0^L \int_{A_s} \epsilon_x^2 dA_s dx \\ &= \frac{EI}{2} \int_0^L z^2 w_{,xx}^2 dx - e_c EA_s \int_0^L \epsilon_x^{pm} w_{,xx} dx + \frac{EA_s}{2} \int_0^L (\epsilon_x^{pm})^2 dx \end{aligned} \quad (51)$$

where I is the moment of inertia about $z = 0$, A_s is the stiffener cross-section area and e_c is the distance from the middle plane of the plate to the centre of area of the stiffener. The integrand in Eq. 51, must be evaluated at the stiffener location $y = y_s$ defined in Fig. 4. By substituting the strain ϵ_x^{pm} from Eq. 5 and the assumed displacement field into Eq.

51, an analytical solution can be derived. In similar manner as for the membrane strain energy of the plate, the strain energy of the stiffener can be separated into a term that is quadratic in the displacement amplitude and a term of a higher order. Then $U^{s,x}$ can be written as

$$U^{s,x} = U^{sL,x} + U^{sNL,x} \quad (52)$$

where $U^{sL,x}$ and $U^{sNL,x}$ give contributions to the linear, incremental stiffness matrix $\mathbf{K}^{sL,x}$, and to the nonlinear, incremental stiffness matrix $\mathbf{K}^{sNL,x}$, respectively. These two matrices is given in Appendix C.

The torsional stiffness of the stiffeners may be accounted for by including the energy contribution (St. Venant torsion) given by

$$U^{sT,x} = \frac{GJ}{2} \int_0^L w_{,xy}^2 dx \quad (53)$$

where J is the torsion constant and $G = E/2(1 + \nu)$. The integrand must be evaluated at the stiffener location $y = y_s$. The strain energy due to torsion of a stiffener is quadratic in the displacement amplitudes. It will therefore give a contribution only to the linear incremental stiffness matrix \mathbf{K}^{sL} . This contribution may be significant in conjunction with torsionally stiff, closed stiffener profiles. In the open stiffener profile examples of the present paper, the torsional stiffener stiffness is neglected. This is normally acceptable for such profiles.

6.2.3 Potential strain energy of a stiffener perpendicular to the free edge

The elastic strain energy of a stiffener perpendicular to free edge is given by

$$\begin{aligned} U^{s,y} &= \frac{E}{2} \int_0^b \int_{A_s} \epsilon_y^2 dA_s dy \\ &= \frac{EI}{2} \int_0^b z^2 w_{,yy}^2 dy - e_c EA_s \int_0^b \epsilon_y^{pm} w_{,yy} dy + \frac{EA_s}{2} \int_0^b (\epsilon_y^{pm})^2 dy \end{aligned} \quad (54)$$

In similar manner as for a stiffener in x -direction, the integrand must be evaluated at the stiffener location $x = x_s$. Further, this contribution can also be separated into a term that is quadratic and a term of a higher order, and then, $U^{s,y}$ can be written as

$$U^{s,y} = U^{sL,y} + U^{sNL,y} \quad (55)$$

where $U^{sL,y}$ and $U^{sNL,y}$ give contributions to the linear, incremental stiffness matrix $\mathbf{K}^{sL,y}$, and to the nonlinear, incremental stiffness matrix $\mathbf{K}^{sNL,y}$, respectively. These two matrices is given in Appendix C.

The torsional stiffness of the stiffeners may be accounted for by including the energy contribution (St. Venant torsion) given by

$$U^{sT,y} = \frac{GJ}{2} \int_0^b w_{,xy}^2 dy \quad (56)$$

The integrand must be evaluated at the stiffener location $x = x_s$.

6.3 Potential energy of external stiffener loads for a stiffener in x -direction

The stiffeners may be end loaded (typical for continuous stiffeners) if the stiffener ends are attached to a surrounding structure. For a continuous longitudinal stiffener, the potential energy of the external loads can be taken according to

$$T^{s,x} = -P_{sx}\Delta u - P_{sx}e_c w_{2,x} + P_{sx}e_c w_{1,x} \quad (57)$$

where P_{sx} is the resultant force (positive in compression) acting on the stiffener. The corresponding rotations at end 1 and end 2 are $w_{1,x}$ and $w_{2,x}$, respectively. The two last terms in Eq. 57 are due to the rotation of the stiffener about the y -axis at the stiffener ends. This expression is similar to an expression for potential energy of external stiffener loads previously given by Brubak and Hellesland [13], and by Steen [34] for a stiffened plate with only one degree of freedom.

7 FINITE ELEMENT MODEL

For verification of the present semi-analytical model, a variety of plate and stiffener dimensions have been considered. Computed results by the present model have been compared with finite element analyses (FEA) using ANSYS [35] in which both plate and stiffeners were modelled using Shell93 elements. In these comparisons, many different results have been studied, including load-displacement curves, displacement plots for a given load, elastic buckling stress limit (ESL) and ultimate strength limit (USL). The ultimate strength (USL) in ANSYS is obtained from fully nonlinear finite element analyses considering both material and geometric nonlinearity. The USL is reached when the external loads reach a maximum value (limit point), and the structure becomes unstable.

In the cases studied, the finite element model is supported in the out-of-plane direction along three edges and with one edge being free. In the same manner as for the proposed model, the plate is subjected to an external axial stress at the two opposite, supported

edges, perpendicular to the free edge. The supported edges are forced to remain straight during deformation, and further, the loaded edges remain parallel. The plate is also supported in the in-plane directions, just enough to prevent rigid body motions. In the cases studied, the ends of the stiffeners are completely free and they are not subjected to any external loads.

In the presented results, the number of degrees of freedom used in ANSYS for a stiffened plate is typically about 15000, which is believed to be a sufficiently large number to ensure satisfactory results. A typical element mesh is shown later (Fig. 20(b)). In comparison, 259 degrees of freedom are used in the proposed model in all cases.

8 LOAD-DISPLACEMENT RESULTS

8.1 Introduction

The present method has been implemented into a FORTRAN computer program, and numerical results obtained by the method have been compared with finite element analyses results from ANSYS for a variety of cases. The load-deflection response for elastic plates computed by the present method is compared with those obtained by large deflection, finite element analysis using ANSYS.

The present load-deflection results are computed without accounting for material yielding, and the response curves are arbitrarily terminated when the external stress S_x reaches the yield stress $f_Y = 235$ MPa. The adopted elastic material properties in each computation are Young's modulus $E = 208000$ MPa and Poisson's ratio $\nu = 0.3$.

The imperfection shape is taken equal to the first eigenmode of the plate also in the nonlinear ANSYS element analyses. For verification purposes, the specified maximum amplitude is taken equal to $w_{0,\text{spec}} = 5$ mm both in the proposed model and ANSYS.

In the analysis by the present model, the total number of degrees of freedom is defined by the number of terms in the assumed displacement fields as given by Eq. 19. The chosen number of terms in each displacement field is

$$M_{wa} = 1, \quad M_{wb} = N_{wb} = 6, \quad M_{ua} = M_{ub} = N_{ub} = M_{va} = M_{vb} = N_{vb} = 10 \quad (58)$$

which results in 259 degrees of freedom. Convergence test have shown that these degrees of freedom give satisfactory results for the plates studied. In these cases, the most severe test case with respect to the number of degrees of freedom is a plate with three regular stiffeners. If plates with more than three stiffeners or plates with clamped edges are analysed, it may be necessary to increase the number of the degrees of freedom.

The present model is also able to analyse plates with simple supports at all four edges, simply by setting $M_{wa} = M_{ua} = M_{va} = 0$. This can be done directly in the FORTRAN computer code. The edge located at $y = b$ will now be simply supported and forced to remain straight during deformation.

As mentioned above, the choice of the number of degrees of freedom will affect the computation time. Another parameter that affects the computation efficiency of the present model, is the incremental step size $\Delta\eta$. A rather small value of $\Delta\eta = 0.04$ is used in the present comparisons with ANSYS results. This value have been found satisfactory in previous investigations [13].

First in this section, unstiffened plates with a free edge is studied, and then, plates with a free edge that is provided with a single edge stiffener. At last, results for plates with three regular stiffeners oriented in the x -direction are presented. In addition, similar comparisons between results by the present model and by ANSYS have been studied for simply supported, unstiffened plates, and good agreement was achieved. Those results are not included in the present report.

8.2 Unstiffened plates with a free edge

Unstiffened plates with a free edge are analysed, and results are presented for four typical cases with plate geometries and dimensions defined in Fig. 7. These plates have intermediate to relatively large slenderness values in order to study cases with rather nonlinear load-displacements curves. These represent relatively severe test cases for the present model.

The displacement shape of the plates computed by ANSYS and the present model is very similar. A typical case is shown in Fig. 8(a) and (b), in which the additional out-of-plane displacements fields w are plotted for Plate 1 subjected to an external stress $S_x = f_y$. Similar comparisons of the in-plane displacement fields in the x - and y -direction have also been made. Again, the results, not included in the present report, are very similar to each other.

In Figs. 9-12, response curves are shown in which the external load S_x is plotted both versus the additional out-of-plane displacement w_{me} at the midlength of the free edge and versus the end shortening Δ_x . The results are given in a non-dimensional form. In the figures, t is the plate thickness and $\epsilon_Y = f_Y/E$ ($= 0.00113$) is the yield strain. The agreement between the response curves computed by the present model (thick solid curve) and by ANSYS (open dots) is good. It can be seen that the curves obtained by the present model is slightly to the non-conservative side, but this discrepancy can be

	L	b	t
Plate 1	1000	1000	12
Plate 2	1000	1000	20
Plate 3	1000	1000	30
Plate 4	2000	1000	30

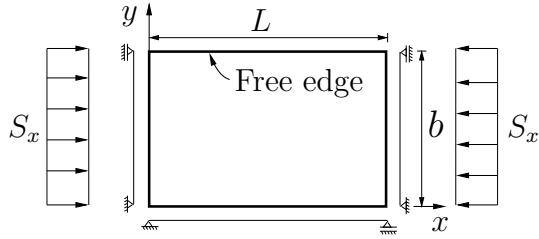


Figure 7: Overview and dimensions [mm] of unstiffened plates with a free edge

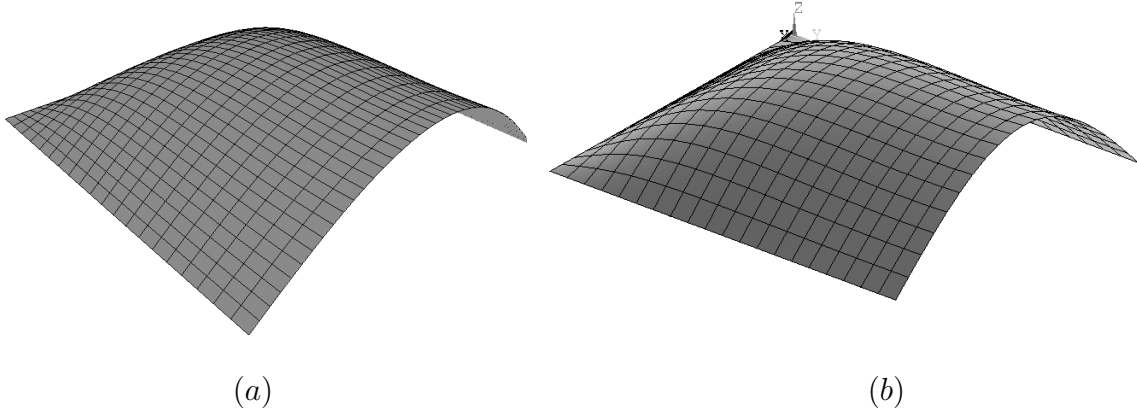


Figure 8: The bending mode of Plate 1 subjected to an external load $S_x = f_Y$ computed (a) by the present model and (b) by ANSYS.

considered as acceptable. The end shortening Δ_x is the reduction of the distance between two opposite edges and it can be considered as a “global displacement”, while the out-of-plane displacement w_{me} is a “local displacement” at the midlength of the free edge. Consequently, it is expected that the agreement between the present model and ANSYS generally is better for the load-shortening curves than for the load-deflection curves. This will especially be true for stiffened plates, as will be seen later.

The elastic buckling stress limit (ESL) computed by the present model is also included in the figures (the dash-dotted lines). This stress gives an indication on the plate slenderness. The corresponding first eigenmode computed by ANSYS and by the present model is quite similar, and as mention before, this mode is used as imperfection. When the

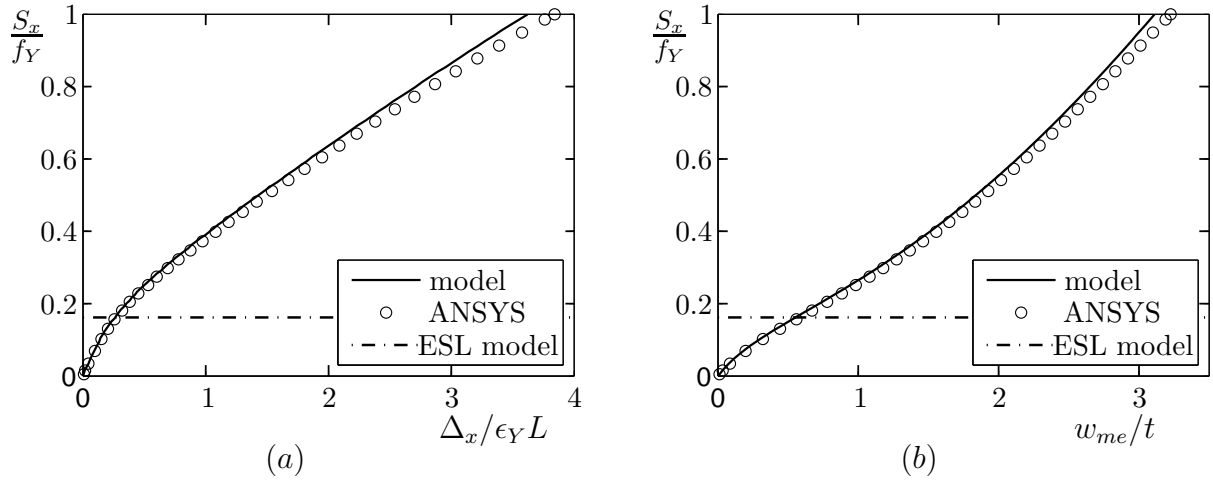


Figure 9: (a) Load-shortening and (b) load-deflection curves of Plate 1 (slender plate) subjected to a uniaxial load S_x .

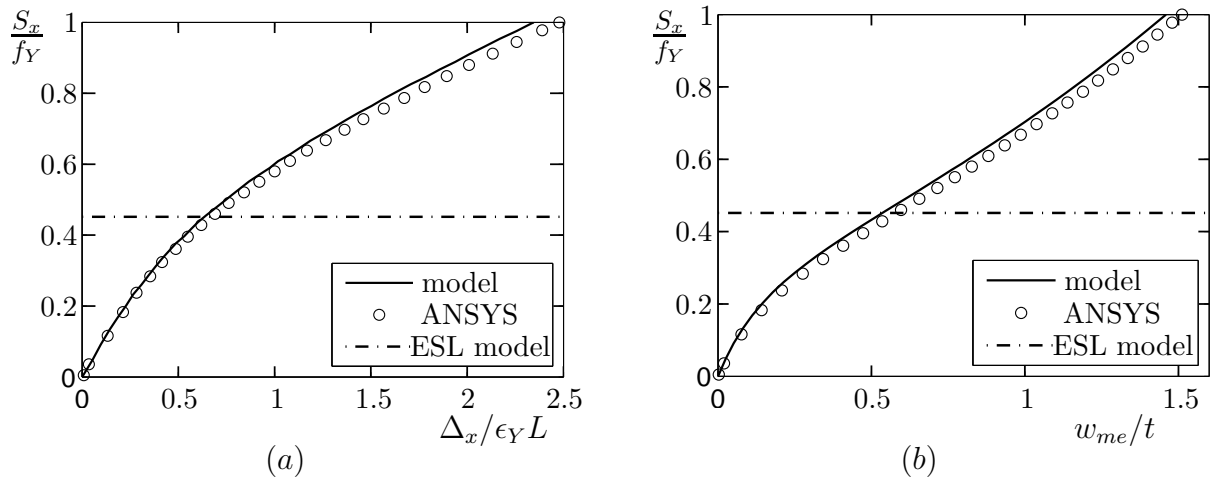


Figure 10: (a) Load-shortening and (b) load-deflection curves of Plate 2 (slender plate) subjected to a uniaxial load S_x .

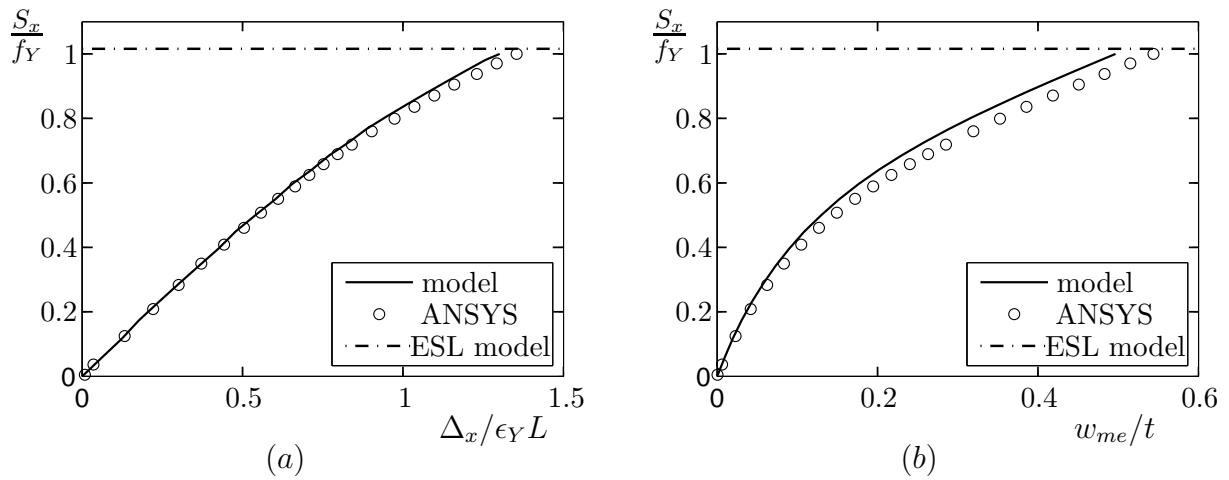


Figure 11: (a) Load-shortening and (b) load-deflection curves of Plate 3 (moderately slender plate) subjected to a uniaxial load S_x .

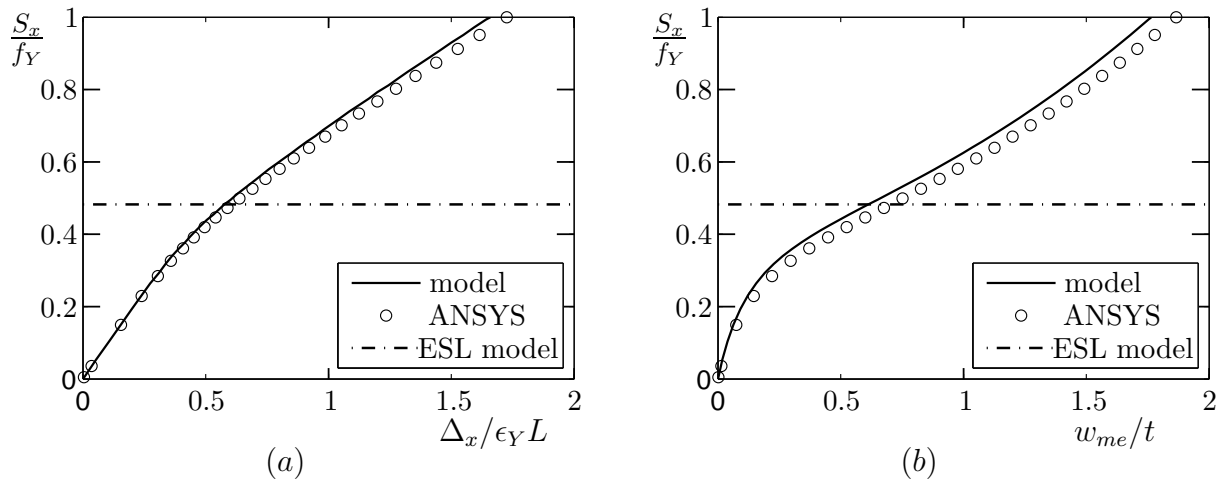


Figure 12: (a) Load-shortening and (b) load-deflection curves of Plate 4 (slender plate) subjected to a uniaxial load S_x .

	L	b	t	h_w	t_w
Plate 5	1000	1000	12	156	10
Plate 6	1000	1000	20	160	10
Plate 7	1000	1000	30	165	10
Plate 8	2000	1000	30	165	10

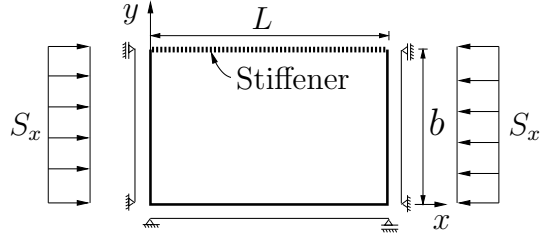


Figure 13: Overview and dimensions [mm] of plates with a free edge provided with an eccentric, flat bar edge stiffener of height $h = 150$ mm.

external stress is close to the elastic buckling stress limit, it can be seen by inspection of the response curves that the plate stiffness is reduced. For external loads far above the ESL value, the plate stiffness will actually increase which can be seen in Fig. 9(b). This behaviour is due to nonlinear membrane effects, and is typical for slender plates with large out-of-plane displacements.

8.3 Stiffened plates with a free edge and an eccentric edge stiffener

The unstiffened plates in the previous section were relatively slender, and such plates may be very unstable. A usual and effective way to increase the strength of a plate with a free edge is to provide it with an edge stiffener. If this edge stiffener is strong enough to prevent out-of-plate displacements, the plate will almost behave as a simply supported plate. In this section, four plates with an edge stiffener will be studied, and results by the present model will be compared with ANSYS results. An overview and plate dimensions are given in Fig. 13. The dimensions of the plates alone are the same as for the unstiffened plates (Plate 1-4) in the previous section, and in addition, an eccentric, flat bar stiffener is attached to the edges in each case. The height of the stiffener web alone is 150 mm.

In Figs. 14-17, similar response curves to that presented above for unstiffened plates are presented for the four plates with an edge stiffener. It can be seen that the agreement between the results of the present model and ANSYS is good. The location of the out-of-plane displacement w_{me} is the same as for the unstiffened plate, but now, this is at the midlength of the edge stiffener and not the midlength of the free edge. Compared to the unstiffened plates, the out-of-plane displacement w_{me} is much smaller than for the corresponding unstiffened plate with the same plate dimensions. The agreement is

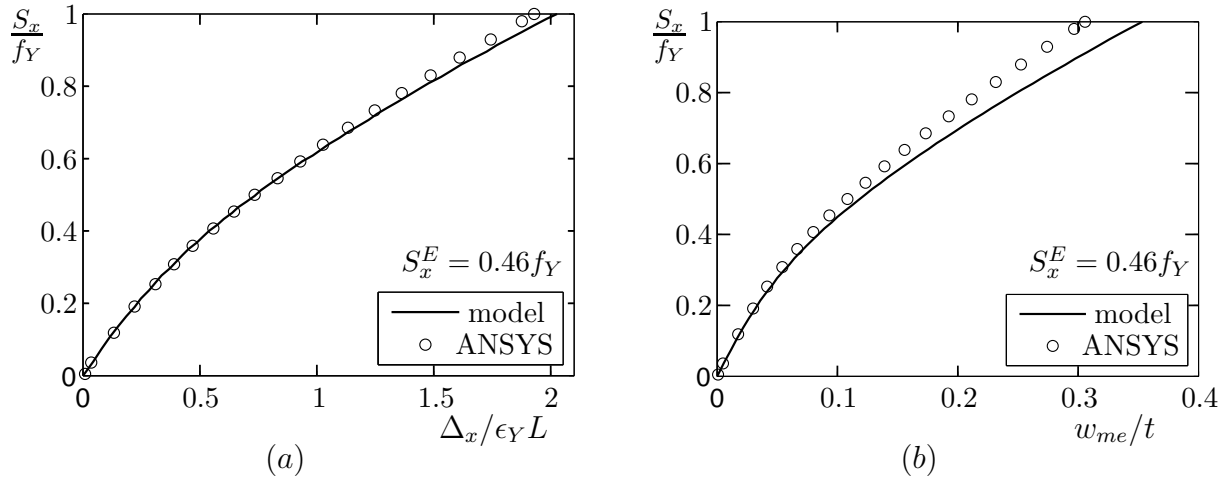


Figure 14: (a) Load-shortening and (b) load-deflection curves of Plate 5 (slender plate) subjected to a uniaxial load S_x .

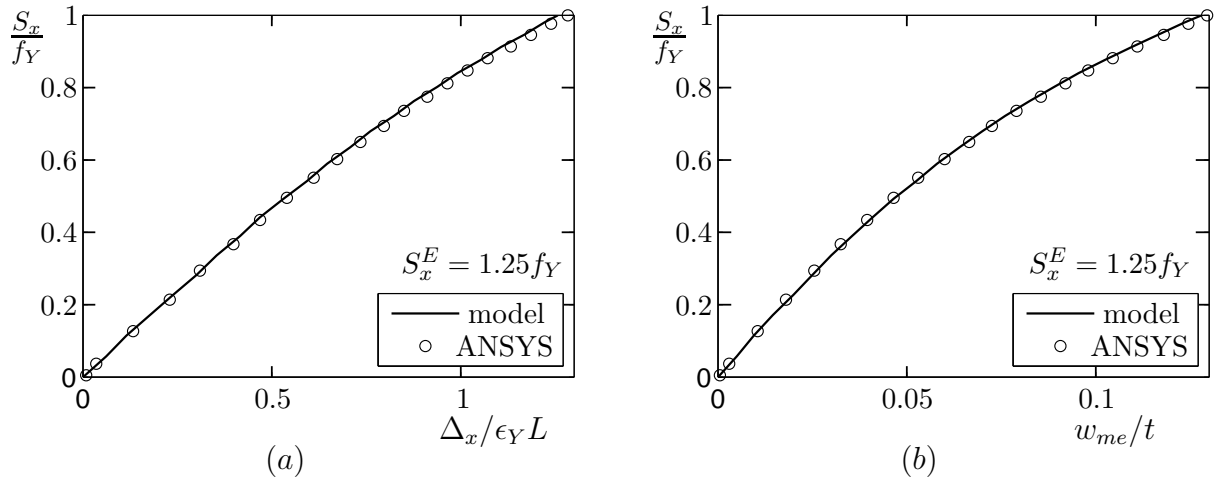


Figure 15: (a) Load-shortening and (b) load-deflection curves of Plate 6 (non-slender plate) subjected to a uniaxial load S_x .

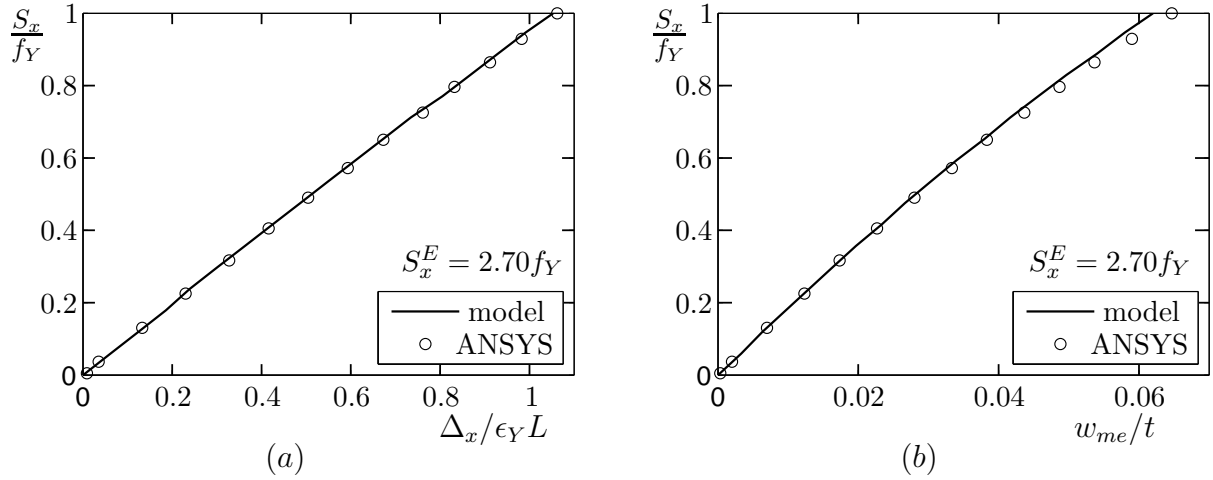


Figure 16: (a) Load-shortening and (b) load-deflection curves of Plate 7 (non-slender plate) subjected to a uniaxial load S_x .

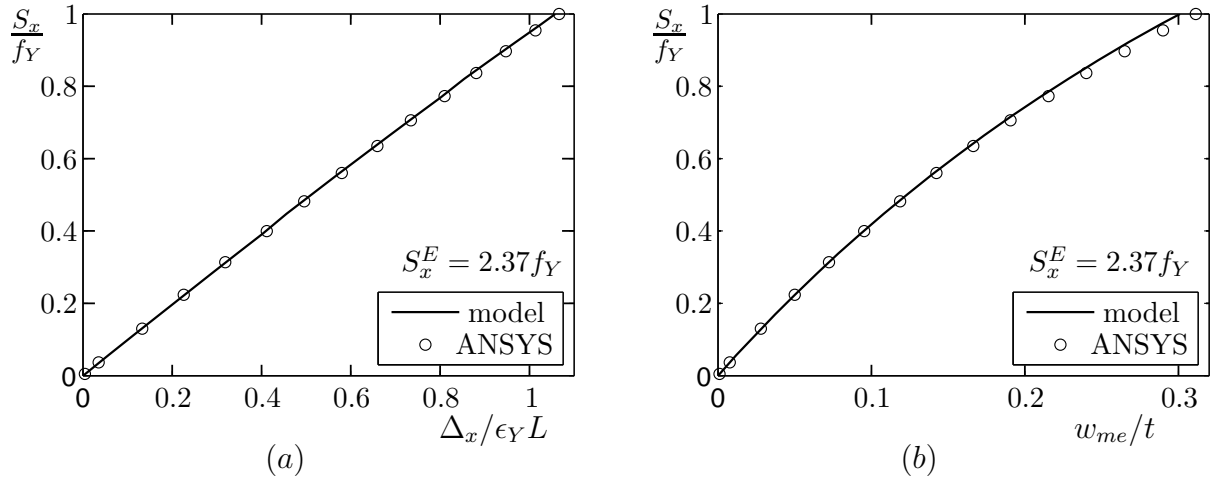


Figure 17: (a) Load-shortening and (b) load-deflection curves of Plate 8 (non-slender plate) subjected to a uniaxial load S_x .

	L	b	t	h_w	t_w
Plate 9	1000	1000	12	56	10
Plate 10	1000	1000	20	60	10
Plate 11	1000	1000	30	65	10
Plate 12	2000	1000	30	65	10
Plate 13	1000	1000	12	106	10

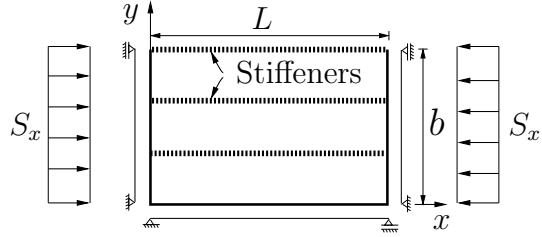


Figure 18: Overview and dimensions [mm] of plates with a free edge provided with three eccentric, flat bar stiffeners of height $h = 50$ mm for Plates 9-12 and $h = 100$ mm for Plate 13.

generally better for the load-shortening curves ($S_x - \Delta_x$) than for the load-deflection curves ($S_x - w_{me}$). As mentioned before, this is to be expected since Δ_x is a global displacement and w_{me} is a local displacement. This is especially the case for plates with relatively strong edge stiffeners preventing out-of-plane displacements, such as for instance, in Fig. 14(b), where there is a clear difference between the model and ANSYS results. For this plate, the out-of-plane displacements are largest in the interior of the plate. The out-of-plane displacement w_{me} at the free edge is therefore not a particularly good parameter for comparison in such cases. The overall out-of-plane bending shape computed by the present model and by ANSYS have been compared, and found to be very similar. The agreement is better for the out-of-plane displacement in the middle of the plate.

The plate stiffness and the elastic buckling stress S_x^E computed by the present model is also included in Figs. 14-17. For these plates, the elastic buckling stress is much larger than that computed for the unstiffened plates. For Plate 6, 7 and 8, the elastic buckling stress is not reached before the external stress becomes equal the yield stress f_Y .

8.4 Stiffened plates with a free edge and three regular, eccentric stiffeners

Similar results to that presented for the plates in the two previous sections, have been obtained for four plates provided with three regular, flat bar stiffeners parallel to the free edge. An overview of the plate and dimensions are given in Fig. 18. Compared to the stiffener dimensions for the plates above with an edge stiffener, the stiffener height for the four first plates (Plate 9-12) is smaller in order to study cases with a global bending

behaviour. In these four cases, the height of the stiffener web alone is 50 mm. In addition, Plate 13 is provided with stiffeners with a web height equal to 100mm, and it will be shown that the bending mode for this plate is a combination of a local and a global bending mode.

Also for these plates, the bending modes of the plates computed by ANSYS and by the present model is very similar. A typical case of a global bending mode is shown in Fig. 19(a) and (b), in which the additional out-of-plane displacements w are plotted for Plate 9 subjected to an external stress $S_x = f_y$. In this case, the stiffeners are not strong enough to prevent large plate deflections along the stiffeners. In Fig. 20(a) and (b), similar plots are shown for Plate 13. It can be seen that the bending mode in this case, is a combination of a global and a local bending mode.

For Plates 9-13, the load-shortening and the load-deflection curves are presented in Figs. 21-25. The agreement between the present model and ANSYS results is seen to be good, except maybe for the load-deflection curve of Plate 13 presented in Fig. 25(b). The displacement w_{me} is a local displacement at the midlength of the edge stiffener, and as mentioned before, it is not expected that the agreement for the load-deflection curves with w_{me} always is good for stiffened plates. However, the results are conservative compared to the ANSYS results, and it is the most conservative case observed by the present model. Moreover, the load-shortening curve computed by the present model for the same plate are in good agreement with ANSYS. The end shortening Δ_x in that response curve can be considered as a global displacement, and consequently better agreement was to be expected.

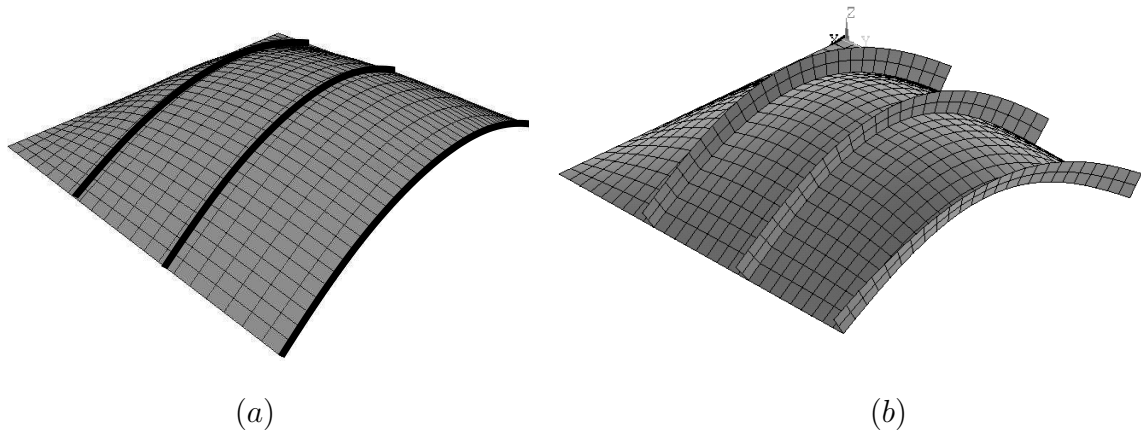


Figure 19: The bending mode of Plate 9 subjected to an external load $S_x = f_Y$ computed (a) by the present model and (b) by ANSYS.

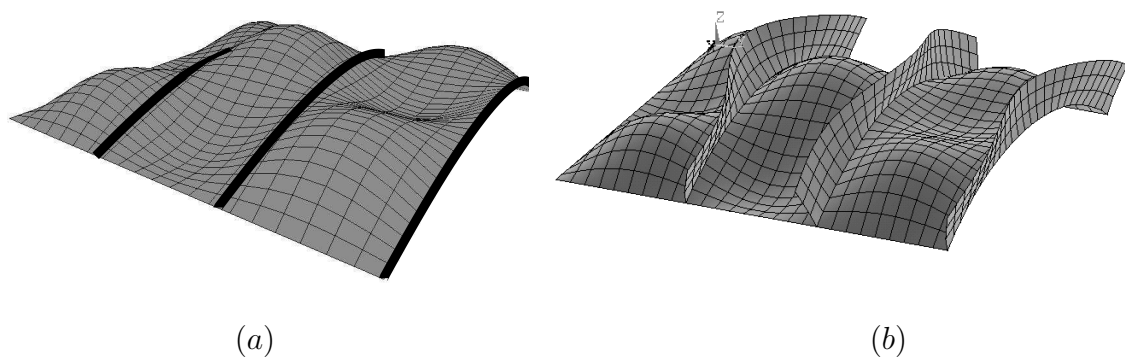


Figure 20: The bending mode of Plate 13 subjected to an external load $S_x = f_Y$ computed (a) by the present model and (b) by ANSYS.

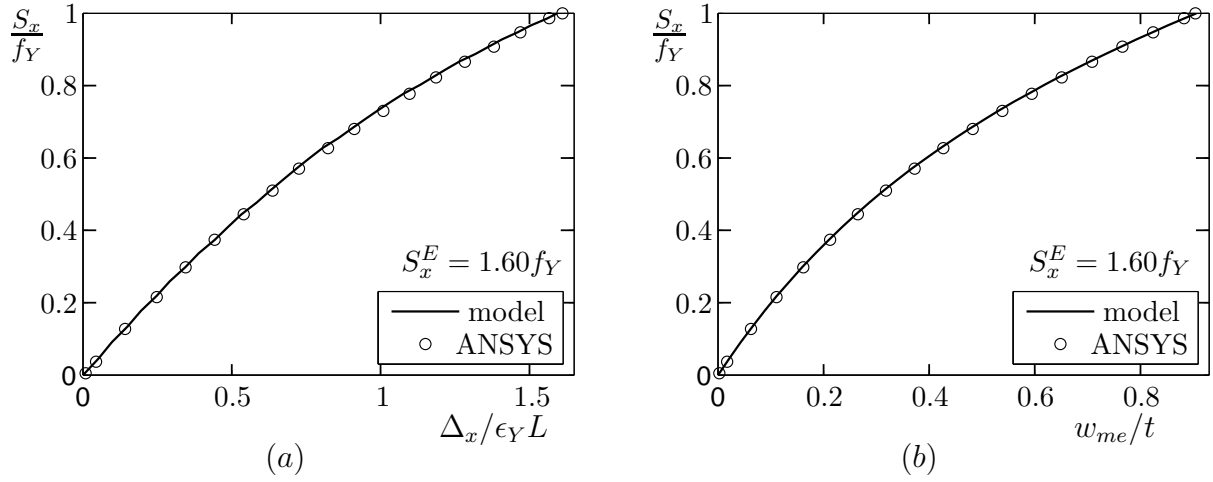


Figure 21: (a) Load-shortening and (b) load-deflection curves of Plate 9 (non-slender plate) subjected to a uniaxial load S_x .

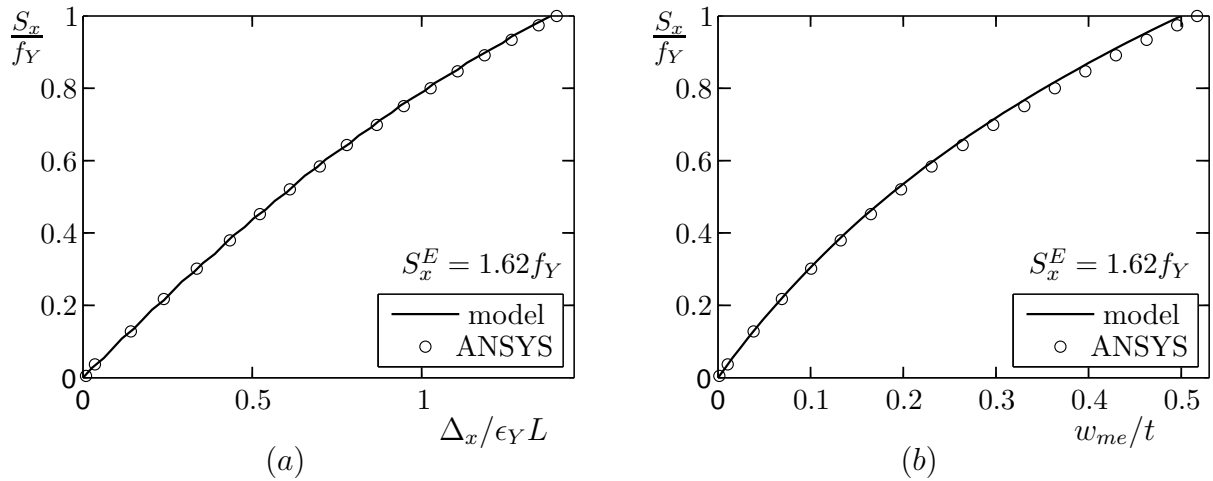


Figure 22: (a) Load-shortening and (b) load-deflection curves of Plate 10 (non-slender plate) subjected to a uniaxial load S_x .

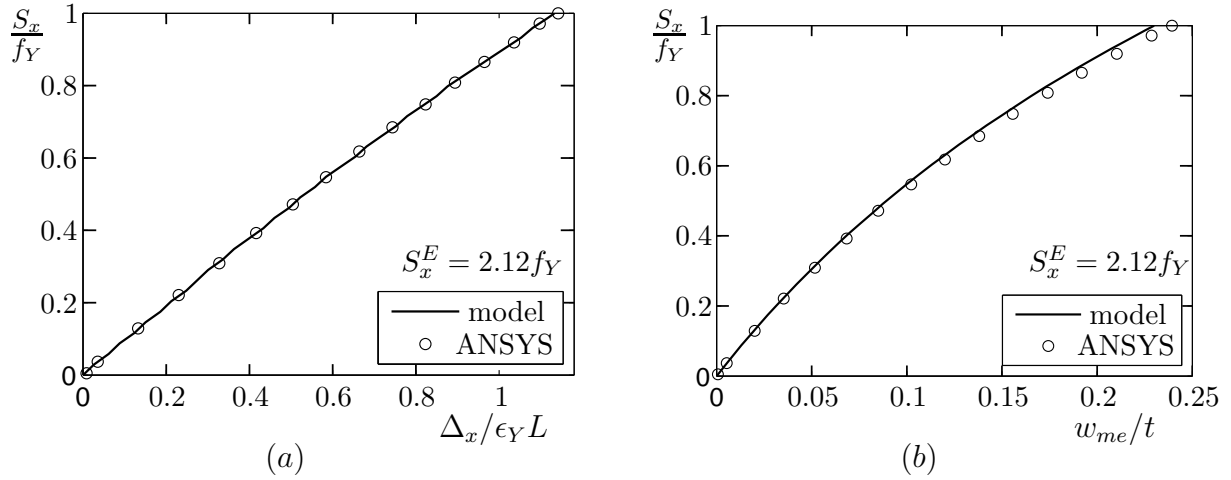


Figure 23: (a) Load-shortening and (b) load-deflection curves of Plate 11 (non-slender plate) subjected to a uniaxial load S_x .

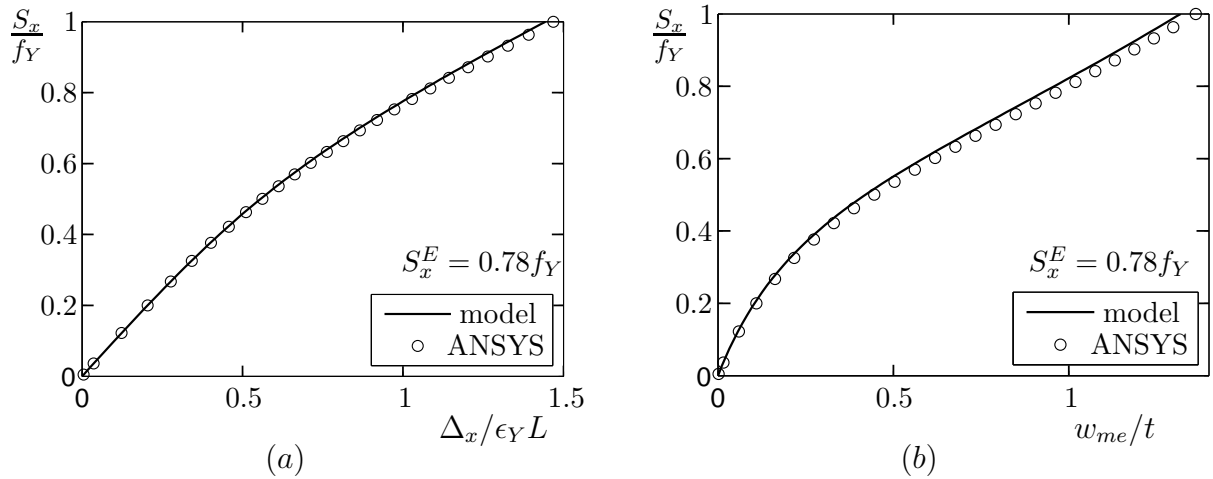


Figure 24: (a) Load-shortening and (b) load-deflection curves of Plate 12 (moderately slender plate) subjected to a uniaxial load S_x .

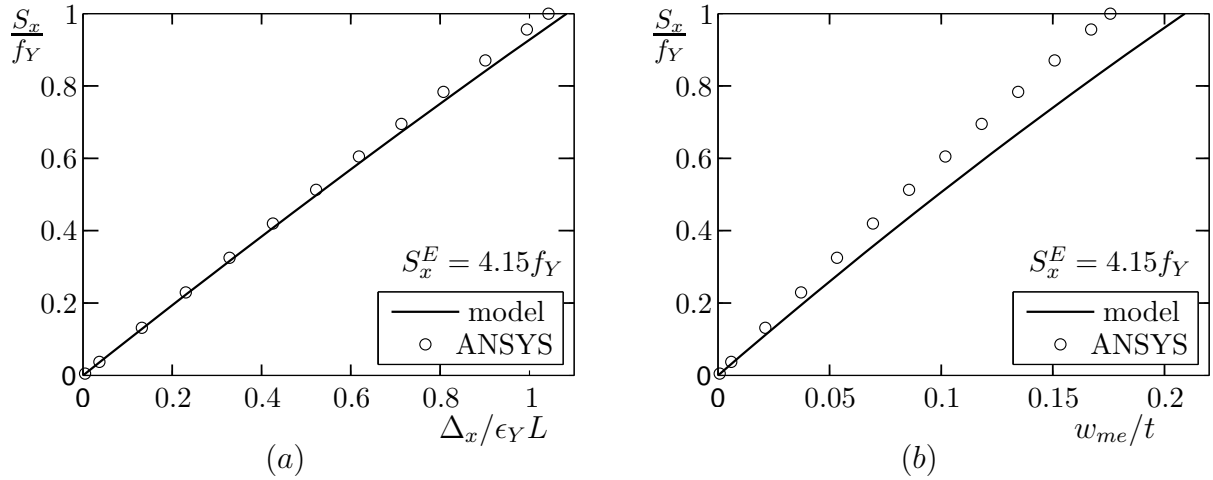


Figure 25: (a) Load-shortening and (b) load-deflection curves of Plate 13 (non-slender plate) subjected to a uniaxial load S_x .

9 CONCLUDING REMARKS

An efficient computational model for large deflection buckling analysis of plates with a free edge is presented. The applicability of the present method is documented for a wide variety of parameters by comparison with finite element analysis results using ANSYS. The model is able to trace the plate response beyond the elastic buckling load. It is able to capture both local and global displacement modes as well as the asymmetric global bending behaviour of plates with eccentric stiffeners.

Due to the computational efficiency of the present model, it is also suited for design optimisation and reliability studies that normally require large number of case studies.

The procedure presented predicts an approximate ultimate strength (USL) by making use of strength criterion in combination with large deflection postbuckling analysis. The method may be somewhat non-conservative for thick plates and plates in global bending, in which cases the bending stress becomes more important. Alternative criteria for such cases are possible topics for further study.

References

- [1] prEN 1993-1-5, Eurocode 3: Design of steel structures. Part 1.5: Plated structural elements, CEN, European Committee for Standardisation, Brussels, 2005

- [2] Det Norske Veritas. DNV Rules for classification of ships, Det Norske Veritas, Høvik, Norway, 2002
- [3] E. Steen and E. Byklum and K.G. Vilming and T.K. Østvold, Computerized buckling models for ultimate strength assessments of stiffened ship hull panels, Proceedings of The Ninth International Symposium on Practical Design of Ships and other Floating Structures, Lübeck-Travemünde, Germany, Sept. 12-17, 2004; 235–242
- [4] J.K. Paik, ALPS/ULSAP user’s manual, Ship Structural Mechanics Lab., Department of Naval Architecture and Ocean Engineering, Pusan National University, Busan, Korea, 2003
- [5] C. Mittelstedt. Stability behaviour of arbitrarily laminated composite plates with free and elastically restrained unloaded edges, International Journal of Mechanical Sciences, 2007; 49(7): 819–833
- [6] S.T. Smith, M.A. Bradford and D.J. Oehlers. Numerical convergence of simple and orthogonal polynomials for the unilateral plate buckling problem using the Rayleigh-Ritz method, International Journal for Numerical Methods in Engineering, 1999; 44(11): 1685–1707
- [7] M. Madhavan and J.S. Davidson. Elastic buckling of I-beam flanges subjected to a linearly varying stress distribution, Journal of Constructional Steel Research, 2007; 63(10): 1373–1383
- [8] M. Madhavan and J.S. Davidson. Buckling of centerline-stiffened plates subjected to uniaxial eccentric compression, Thin-Walled Structures, 2008; 43(5): 1264–1276
- [9] P. Qiao and L. Shan. Explicit local buckling analysis and design of fiber-reinforced plastic composite structural shapes, Composite Structures, 2005; 70(4): 468–483
- [10] C. Yu and B.W. Schafer. Effect of longitudinal stress gradients on elastic buckling of thin plates, Journal of Engineering Mechanics, ASCE, 2007; 133(4): 452–463
- [11] E. Steen. Application of the perturbation method to plate buckling problems, Research Report in Mechanics, No. 98-1, Mechanics Division, Dept. of Mathematics, University of Oslo, Norway, 1998, 60 pp.
- [12] E. Byklum and J. Amdahl. A simplified method for elastic large deflection analysis of plates and stiffened panels due to local buckling, Thin-Walled Structures, 2000; 40(11): 925–953

- [13] Lars Brubak and Jostein Helleland. Semi-analytical postbuckling and strength analysis of arbitrarily stiffened plates in local and global bending, *Thin-Walled Structures*, 2007; 45(6), 620–633
- [14] prEN 1993-1-1, Eurocode 3: Design of steel structures. Part 1.1: General rules and rules for buildings, CEN, European Committee for Standardisation, Brussels, 2003
- [15] Det Norske Veritas. Recommended practice DNV-RP-C201, Buckling strength of plated structures, Høvik, Norway, 2002
- [16] K. Marguerre. Zur theorie der gekrümmten platte grosser formänderung, *Proceedings of The 5th International Congress for Applied Mechanics*, 1938; 93–101
- [17] H.R. Ovesy, J. Loughlan, H. Assae. The compressive post-local bucking behaviour of thin plates using a semi-energy finite strip approach, *Thin-Walled Structures*, 2004; 42(3): 449–474
- [18] L. Brubak, J. Helleland and E. Steen. Semi-analytical buckling strength analysis of plates with arbitrary stiffener arrangements, *Journal of Constructional Steel Research*, 2007, 63(4): 532–543
- [19] D.O. Brush and B.O. Almroth. Buckling of bars, plates and shells, McGraw-Hill Book Company, 1975
- [20] Z.P. Bažant and L. Cedolin. *Stability of structures*, Oxford University Press, 1991
- [21] Lars Brubak and Jostein Helleland. Approximate buckling strength analysis of arbitrarily stiffened, stepped plates, *Engineering Structures*, 2007; 29(9): 2321–2333
- [22] H.R. Ovesy, J. Loughlan, S.A.M. GhannadPour. Geometric non-linear analysis of channel sections under end shortening, using different versions of the finite strip method *Computers and Structures*, 2006; 84(13-14): 855–872
- [23] L. Brubak. Semi-analytical buckling strength analysis of plates with constant or varying thickness and arbitrarily oriented stiffeners. Research report in mechanics, No. 05-6, Mechanics Division, Dept. of Mathematics, University of Oslo, Norway, 2005, 65 pp.
- [24] H.M. Eiding. Knekning av avstivede plater med variable randbetingelser (Buckling of stiffened plates with various boundary conditions), Mechanics Division, Department of Mathematics, University of Oslo, Norway, 2007, 87 pp.

- [25] L.K. Seah and J. Rhodes. Simplified buckling analysis of plate with compound edge stiffeners, *Journal of Engineering Mechanics*, 1993; 119(1): 19–37
- [26] O.K. Bedair. The elastic behaviour of multi-stiffened plates under uniform compression, *Thin-Walled Structures*, 1997; 27(4): 311–335
- [27] E. Byklum. Ultimate strength analysis of stiffened steel and aluminium panels using semi-analytical methods, Dr. Ing. thesis, Norwegian University of Science and Technology, Trondheim, Norway, 2002
- [28] J.K. Paik and M.S. Lee, A Semi-analytical method for the elastic-plastic large deflection analysis of stiffened panels under combined biaxial compression/tension, biaxial in-plate bending, edge shear, and lateral pressure loads, *Thin-Walled Structures*, 2005; 43(3): 375–410
- [29] E. Byklum, E. Steen and J. Amdahl. A semi-analytical model for global buckling and postbuckling analysis of stiffened panels, *Thin-Walled Structures*, 2004; 42(5): 701–717
- [30] E. Steen, E. Byklum and J. Helleland. Elastic postbuckling stiffness of biaxially compressed rectangular plates, *Engineering Structures*, 2008; 30(10): 2631–2643
- [31] E. Riks. An incremental approach to the solution of snapping and buckling problems, *International Journal of Solids and Structures*, 1979; 15: 529–551
- [32] E. Kreyszig. *Advanced Engineering Mathematics*, 7th ed., John Wiley & Sons, Inc., 1993
- [33] Lars Brubak and Jostein Helleland. Strength criteria in semi-analytical, large deflection analysis of stiffened plates in local and global bending, accepted for publication in *Thin-Walled Structures*, 2007
- [34] E. Steen. Elastic buckling and postbuckling of eccentrically stiffened plates, *International Journal of Solids and Structures*, 1989; 25(7): 751–768
- [35] ANSYS Inc., *ANSYS Documentation 9.0*, Southpointe, Canonsburg, PA, 2004.

A Subdivision of the matrices and vectors

$$\mathbf{K}_{\mathbf{uu}} = \begin{bmatrix} \mathbf{K}_{uaua} & \mathbf{K}_{uaub} & \mathbf{K}_{uauc} \\ \mathbf{K}_{ubua} & \mathbf{K}_{ubub} & \mathbf{K}_{ubuc} \\ \mathbf{K}_{ucua} & \mathbf{K}_{ucub} & \mathbf{K}_{ucuc} \end{bmatrix}, \quad \mathbf{K}_{\mathbf{vv}} = \begin{bmatrix} \mathbf{K}_{vava} & \mathbf{K}_{vavb} & \mathbf{K}_{vavc} \\ \mathbf{K}_{vbva} & \mathbf{K}_{vbnb} & \mathbf{K}_{vbvc} \\ \mathbf{K}_{vcva} & \mathbf{K}_{vcvb} & \mathbf{K}_{vcvc} \end{bmatrix} \quad (59)$$

$$\mathbf{K}_{\mathbf{ww}} = \begin{bmatrix} \mathbf{K}_{wawa} & \mathbf{K}_{wawb} \\ \mathbf{K}_{wbwa} & \mathbf{K}_{wbwb} \end{bmatrix}, \quad \mathbf{K}_{\mathbf{uv}} = \mathbf{K}_{\mathbf{vu}}^T = \begin{bmatrix} \mathbf{K}_{uava} & \mathbf{K}_{uavb} & \mathbf{K}_{uavc} \\ \mathbf{K}_{ubva} & \mathbf{K}_{ubvb} & \mathbf{K}_{ubvc} \\ \mathbf{K}_{ucva} & \mathbf{K}_{ucvb} & \mathbf{K}_{ucvc} \end{bmatrix} \quad (60)$$

$$\mathbf{K}_{\mathbf{uw}} = \mathbf{K}_{\mathbf{wu}}^T = \begin{bmatrix} \mathbf{K}_{uawa} & \mathbf{K}_{uawb} \\ \mathbf{K}_{ubwa} & \mathbf{K}_{ubwb} \\ \mathbf{K}_{ucwa} & \mathbf{K}_{ucwb} \end{bmatrix}, \quad \mathbf{K}_{\mathbf{vw}} = \mathbf{K}_{\mathbf{wv}}^T = \begin{bmatrix} \mathbf{K}_{vawa} & \mathbf{K}_{vawb} \\ \mathbf{K}_{vbwa} & \mathbf{K}_{vbwb} \\ \mathbf{K}_{vcwa} & \mathbf{K}_{vcwb} \end{bmatrix} \quad (61)$$

$$\mathbf{G}_{\mathbf{u}} = [\mathbf{G}_{ua}, \mathbf{G}_{ub}, \mathbf{G}_{uc}]^T, \quad \mathbf{G}_{\mathbf{v}} = [\mathbf{G}_{va}, \mathbf{G}_{vb}, \mathbf{G}_{vc}]^T, \quad \mathbf{G}_{\mathbf{w}} = [\mathbf{G}_{wa}, \mathbf{G}_{wb}, \mathbf{G}_{wc}]^T \quad (62)$$

$$\mathbf{u} = [\mathbf{u}^a, \mathbf{u}^b, \mathbf{u}^c]^T, \quad \mathbf{v} = [\mathbf{v}^a, \mathbf{v}^b, \mathbf{v}^c]^T, \quad \mathbf{w} = [\mathbf{w}^a, \mathbf{w}^b]^T \quad (63)$$

B Rate form of energy contributions of the plate

B.1 \mathbf{K}^{pb} -matrices

B.1.1 \mathbf{K}_{ww}^{pb} -matrix

$$\mathbf{K}_{wawa}^{pb} \dot{\mathbf{w}}^a = \frac{\partial^2 U^{pb}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a = D \left[\left(\frac{f\pi}{L} \right)^4 \frac{Lb}{6} + (1-\nu) \left(\frac{f\pi}{L} \right)^2 \frac{L}{b} \right] \dot{w}_f^a \quad (64)$$

$$\mathbf{K}_{wawb}^{pb} \dot{\mathbf{w}}^b = \frac{\partial^2 U^{pb}}{\partial w_f^a \partial w_{pq}^{pb}} \dot{w}_{pq}^b = \sum_{q=1}^{N_{wb}} D \left[\left(\frac{f\pi}{L} \right)^4 \frac{L}{2} I_q^{ys} + \nu \left(\frac{f\pi}{L} \right)^2 \left(\frac{q\pi}{L} \right)^2 \frac{L}{2} I_q^{ys} \right] \dot{w}_{fq}^b \quad (65)$$

$$\mathbf{K}_{wbwa}^{pb} \dot{\mathbf{w}}^b = \frac{\partial^2 U^{pb}}{\partial w_{fg}^b \partial w_p^a} \dot{w}_p^a = D \left[\left(\frac{f\pi}{L} \right)^4 \frac{L}{2} I_g^{ys} + \nu \left(\frac{f\pi}{L} \right)^2 \left(\frac{g\pi}{L} \right)^2 \frac{L}{2} I_g^{ys} \right] \dot{w}_f^a \quad (66)$$

$$\mathbf{K}_{wbwb}^{pb} \dot{\mathbf{w}}^b = \frac{\partial^2 U^{pb}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b = D \frac{Lb}{4} \left[\left(\frac{f\pi}{L} \right)^2 + \left(\frac{g\pi}{L} \right)^2 \right] \dot{w}_{fg}^b \quad (67)$$

B.2 \mathbf{K}^{pm} -matrices

B.2.1 \mathbf{K}_{uu}^{pmL} -matrix

$$\mathbf{K}_{uaua}^{pmL} \dot{\mathbf{u}}^a = \frac{\partial^2 U^{pmL}}{\partial u_f^a \partial u_p^a} \dot{u}_p^a = C \left[\left(\frac{f\pi}{L} \right)^2 \frac{b}{3} \frac{L}{2} + \frac{1-\nu}{2} \frac{1}{b} \frac{L}{2} \right] \dot{u}_f^a \quad (68)$$

$$\mathbf{K}_{ubub}^{pmL} \dot{\mathbf{u}}^b = \frac{\partial^2 U^{pmL}}{\partial u_{fg}^b \partial u_{pq}^b} \dot{u}_{pq}^b = C \left[\left(\frac{f\pi}{L} \right)^2 + \frac{1-\nu}{2} \left(\frac{g\pi}{b} \right)^2 \right] \frac{Lb}{4} \dot{u}_{fg}^b \quad (69)$$

$$\mathbf{K}_{ucuc}^{pmL} \dot{\mathbf{u}}^c = \frac{\partial^2 U^{pmL}}{\partial u^c \partial u^c} \dot{u}^c = C \frac{b}{L} \dot{u}^c \quad (70)$$

$$\mathbf{K}_{uaub}^{pmL} \dot{\mathbf{u}}^b = \frac{\partial^2 U^{pmL}}{\partial u_f^a \partial u_{pq}^b} \dot{u}_{pq}^b = \sum_{q=1}^{N_{ub}} C \left(\frac{f\pi}{L} \right)^2 \frac{L}{2} I_q^{ys} \dot{u}_{fq}^b \quad (71)$$

$$\mathbf{K}_{ubua}^{pmL} \dot{\mathbf{u}}^a = \frac{\partial^2 U^{pmL}}{\partial u_{fg}^b \partial u_p^a} \dot{u}_p^a = C \left(\frac{f\pi}{L} \right)^2 \frac{L}{2} I_g^{ys} \dot{u}_f^a \quad (72)$$

$$\mathbf{K}_{uauc}^{pmL} = 0, \quad \mathbf{K}_{ucua}^{pmL} = 0, \quad \mathbf{K}_{ubuc}^{pmL} = 0, \quad \mathbf{K}_{ucub}^{pmL} = 0 \quad (73)$$

B.2.2 \mathbf{K}_{uv}^{pmL} -matrix

$$\mathbf{K}_{uava}^{pmL} \dot{\mathbf{v}}^a = \frac{\partial^2 U^{pmL}}{\partial u_f^a \partial v_p^a} \dot{v}_p^a = C \frac{f\pi}{L} \frac{L}{4} \frac{3\nu-1}{2} \dot{v}_f^a \quad (74)$$

$$\mathbf{K}_{uavb}^{pmL} \dot{\mathbf{v}}^b = \frac{\partial^2 U^{pmL}}{\partial u_f^a \partial v_{pq}^b} \dot{v}_{pq}^b = \sum_{p=1}^{M_{vb}} \sum_{q=1}^{N_{vb}} C \left[\nu \frac{f\pi}{L} \frac{q\pi}{b} I_q^{yc} I_{fp}^{cs} + \frac{1-\nu}{2} \frac{p\pi}{L} \frac{1}{b} I_q^s I_{pf}^{cs} \right] \dot{v}_{pq}^b \quad (75)$$

$$\mathbf{K}_{ubva}^{pmL} \dot{\mathbf{v}}^a = \frac{\partial^2 U^{pmL}}{\partial u_{fg}^b \partial v_p^a} \dot{v}_p^a = C \left[\nu \frac{f\pi}{L} \frac{1}{b} \frac{L}{2} I_g^s - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{g\pi}{b} \frac{L}{2} I_g^{yc} \right] \dot{v}_f^a \quad (76)$$

$$\mathbf{K}_{ubvb}^{pmL} \dot{\mathbf{v}}^b = \frac{\partial^2 U^{pmL}}{\partial u_{fg}^b \partial v_{pq}^b} \dot{v}_{pq}^b = \sum_{p=1}^{M_{vb}} \sum_{q=1}^{N_{vb}} C \left[\nu \frac{f\pi}{L} \frac{q\pi}{b} I_{fp}^{cs} I_{qg}^{cs} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} I_{pf}^{cs} I_{gq}^{cs} \right] \dot{v}_{pq}^b \quad (77)$$

$$\mathbf{K}_{ucvc}^{pmL} \dot{\mathbf{v}}^c = \frac{\partial^2 U^{pmL}}{\partial u^c \partial v^c} \dot{v}^c = C \nu \dot{v}^c \quad (78)$$

$$\mathbf{K}_{uavc}^{pmL} = 0, \quad \mathbf{K}_{vcua}^{pmL} = 0, \quad \mathbf{K}_{ubvc}^{pmL} = 0, \quad \mathbf{K}_{vcub}^{pmL} = 0 \quad (79)$$

B.2.3 \mathbf{K}_{vv}^{pmL} -matrix

$$\mathbf{K}_{vava}^{pmL} \dot{\mathbf{v}}^a = \frac{\partial^2 U^{pmL}}{\partial v_f^a \partial v_p^a} \dot{v}_p^a = C \left[\frac{1}{b} + \frac{1-\nu}{2} \left(\frac{f\pi}{L} \right)^2 \frac{b}{3} \right] \frac{L}{2} \dot{v}_f^a \quad (80)$$

$$\mathbf{K}_{vbnb}^{pmL} \dot{\mathbf{v}}^b = \frac{\partial^2 U^{pmL}}{\partial v_{fg}^b \partial v_{pq}^b} \dot{v}_{pq}^b = C \left[\left(\frac{g\pi}{b} \right)^2 + \frac{1-\nu}{2} \left(\frac{f\pi}{L} \right)^2 \right] \frac{Lb}{4} \dot{v}_{fg}^b \quad (81)$$

$$\mathbf{K}_{vcvc}^{pmL} \dot{\mathbf{v}}^c = \frac{\partial^2 U^{pmL}}{\partial v^c \partial v^c} \dot{v}^c = C \frac{L}{b} \dot{v}^c \quad (82)$$

$$\mathbf{K}_{vavb}^{pmL} \dot{\mathbf{v}}^a = \frac{\partial^2 U^{pmL}}{\partial v_f^a \partial v_{pq}^b} \dot{v}_{pq}^b = - \sum_{p=1}^{M_{vb}} \sum_{q=1}^{N_{vb}} C \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} I_q^{ys} I_{pf}^{cs} \dot{v}_{pq}^b \quad (83)$$

$$\mathbf{K}_{vbva}^{pmL} \dot{\mathbf{v}}^a = \frac{\partial^2 U^{pmL}}{\partial v_{fg}^b \partial v_p^a} \dot{v}_p^a = - \sum_{p=1}^{M_{va}} C \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} I_g^{ys} I_{fp}^{cs} \dot{v}_p^a \quad (84)$$

$$\mathbf{K}_{vavc}^{pmL} = 0, \quad \mathbf{K}_{vcva}^{pmL} = 0, \quad \mathbf{K}_{vbbc}^{pmL} = 0, \quad \mathbf{K}_{vcvb}^{pmL} = 0 \quad (85)$$

B.2.4 \mathbf{K}_{uw}^{pmL} -matrix

$$\begin{aligned} \mathbf{K}_{uawa}^{pmL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmL}}{\partial u_f^a \partial w_p^a} \dot{w}_p^a \\ &= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0m}^a C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{b}{4} I_{fpm}^{ccc} + \frac{1-\nu}{2} \left(\frac{1}{b} \right)^2 \frac{b}{2} \frac{m\pi}{L} I_{mfp}^{css} \right. \\ &\quad \left. + \nu \frac{f\pi}{L} \left(\frac{1}{b} \right)^2 \frac{b}{2} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{p\pi}{L} \left(\frac{1}{b} \right)^2 \frac{b}{2} I_{pjm}^{css} \right] \dot{w}_p^a \\ &+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_n^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \left(\frac{1}{b} \right)^2 \frac{m\pi}{L} I_n^{ss} I_{mfp}^{css} \right. \\ &\quad \left. + \nu \frac{f\pi}{L} \frac{1}{b} \frac{n\pi}{b} I_n^{yc} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{p\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_n^{yc} I_{pjm}^{css} \right] \dot{w}_p^a \end{aligned} \quad (86)$$

$$\begin{aligned}
\mathbf{K}_{uawb}^{pmL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmL}}{\partial u_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_q^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{q\pi}{b} \frac{m\pi}{L} \frac{1}{b} I_q^{yc} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_q^{yc} I_{fpm}^{css} + \frac{1-\nu}{2} \left(\frac{1}{b} \right)^2 \frac{p\pi}{L} I_q^{cs} I_{pfm}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_{qn}^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{1}{b} \frac{q\pi}{b} \frac{m\pi}{L} I_{qn}^{cs} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{q\pi}{b} \frac{n\pi}{b} I_{qn}^{yc} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{p\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_{nq}^{cs} I_{pfm}^{css} \right] \dot{w}_{pq}^b
\end{aligned} \tag{87}$$

$$\begin{aligned}
\mathbf{K}_{ubwa}^{pmL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmL}}{\partial u_{fg}^b \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} w_{0m}^a C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_g^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{m\pi}{L} \frac{1}{b} I_g^{yc} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \left(\frac{1}{b} \right)^2 I_g^{cs} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_g^{yc} I_{pfm}^{css} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_{qn}^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{1}{b} \frac{g\pi}{b} \frac{m\pi}{L} I_{gn}^{cs} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_{nq}^{cs} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} \frac{n\pi}{b} I_{gn}^{yc} I_{pfm}^{css} \right] \dot{w}_p^a
\end{aligned} \tag{88}$$

$$\begin{aligned}
\mathbf{K}_{ubwb}^{pmL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmL}}{\partial u_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_{gq}^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{q\pi}{b} \frac{m\pi}{L} I_{gq}^{yc} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{gq}^{cs} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_{gq}^{cs} I_{pfm}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_{fpm}^{ccc} I_{gqn}^{sss} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{q\pi}{b} \frac{m\pi}{L} I_{mfp}^{css} I_{ngq}^{scc} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{q\pi}{b} \frac{n\pi}{b} I_{fpm}^{css} I_{gqn}^{scc} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} \frac{n\pi}{b} I_{qgn}^{scc} I_{pfm}^{css} \right] \dot{w}_{pq}^b
\end{aligned} \tag{89}$$

$$\begin{aligned}
\mathbf{K}_{ucwa}^{pmL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmL}}{\partial u^c \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} w_{0p}^a C \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 \frac{Lb}{6} + \nu \frac{1}{L} \frac{1}{b} \frac{L}{2} \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wab}} \sum_{n=1}^{N_{wb}} w_{0pn}^b C \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 I_n^{ys} \frac{L}{2} \right] \dot{w}_p^a
\end{aligned} \tag{90}$$

$$\begin{aligned}
\mathbf{K}_{ucwb}^{pmL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmL}}{\partial u^c \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wab}} \sum_{q=1}^{N_{wb}} w_{0p}^a C \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 I_q^{ys} \frac{L}{2} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} w_{0pq}^b C \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 + \nu \frac{1}{L} \left(\frac{q\pi}{b} \right)^2 \right] \frac{Lb}{4} \dot{w}_{pq}^b
\end{aligned} \tag{91}$$

B.2.5 \mathbf{K}_{vw}^{pmL} -matrix

$$\begin{aligned}
\mathbf{K}_{vawa}^{pmL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmL}}{\partial v_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0m}^a C \left[\left(\frac{1}{b} \right)^2 I_{fpm}^{css} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} \frac{b}{3} I_{pfm}^{css} \right. \\
&\quad \left. + \nu \frac{p\pi}{L} \frac{m\pi}{L} \frac{1}{b} \frac{b}{3} I_{fpm}^{ccc} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} \frac{b}{3} I_{mfp}^{css} \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b C \left[\nu \frac{p\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_n^{ys} I_{fpm}^{ccc} \right. \\
&\quad \left. - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} I_n^{yyc} I_{pfm}^{css} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_n^{ys} I_{mfp}^{css} \right] \dot{w}_p^a
\end{aligned} \tag{92}$$

$$\begin{aligned}
\mathbf{K}_{vawb}^{pmL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmL}}{\partial v_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a C \left[\nu \frac{p\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_q^{ys} I_{fpm}^{ccc} \right. \\
&\quad \left. - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_q^{ys} I_{pfm}^{css} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{m\pi}{L} I_q^{yyc} I_{mfp}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b C \left[\frac{q\pi}{b} \frac{n\pi}{b} \frac{1}{b} \frac{1}{2} \delta_{q,n} I_{fpm}^{css} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} I_{nq}^{ycs} I_{pfm}^{css} \right. \\
&\quad \left. + \nu \frac{p\pi}{L} \frac{m\pi}{L} \frac{1}{b} \frac{1}{2} \delta_{q,n} I_{fpm}^{ccc} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{m\pi}{L} I_{qn}^{ycs} I_{mfp}^{css} \right] \dot{w}_{pq}^b
\end{aligned} \tag{93}$$

$$\begin{aligned}
\mathbf{K}_{vbwa}^{pmL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmL}}{\partial v_{fg}^b \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0m}^a C \left[\frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_g^{ys} I_{mfp}^{scc} \right. \\
&\quad \left. + \nu \frac{g\pi}{b} \frac{p\pi}{L} \frac{m\pi}{L} I_g^{yyc} I_{fpm}^{scc} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_g^{ys} I_{pfm}^{scc} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b C \left[\frac{g\pi}{b} \frac{n\pi}{b} \frac{1}{b} \frac{1}{2} \delta_{g,n} I_{fpm}^{sss} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} I_{ng}^{ycs} I_{mfp}^{scc} \right. \\
&\quad \left. + \nu \frac{g\pi}{b} \frac{p\pi}{L} \frac{m\pi}{L} I_{gn}^{ycs} I_{fpm}^{scc} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} \frac{1}{2} \delta_{g,n} I_{pfm}^{scc} \right] \dot{w}_p^a
\end{aligned} \tag{94}$$

$$\begin{aligned}
\mathbf{K}_{vbwb}^{pmL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmL}}{\partial v_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a C \left[\frac{g\pi}{b} \frac{q\pi}{b} \frac{1}{b} \frac{1}{2} \delta_{g,n} I_{fpm}^{sss} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} \frac{1}{2} \delta_{g,n} I_{mfp}^{scc} \right. \\
&\quad \left. + \nu \frac{g\pi}{b} \frac{p\pi}{L} \frac{m\pi}{L} I_{gq}^{ycs} I_{fpm}^{scc} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{m\pi}{L} I_{qg}^{ycs} I_{pfm}^{scc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b C \left[\frac{g\pi}{b} \frac{q\pi}{b} \frac{n\pi}{b} I_{fpm}^{sss} I_{gqn}^{ccc} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} I_{mfp}^{scc} I_{ngq}^{css} \right. \\
&\quad \left. + \nu \frac{g\pi}{b} \frac{p\pi}{L} \frac{m\pi}{L} I_{fpm}^{scc} I_{gqn}^{css} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{m\pi}{L} I_{pfm}^{scc} I_{qgn}^{css} \right] \dot{w}_{pq}^b
\end{aligned} \tag{95}$$

$$\begin{aligned}
\mathbf{K}_{vcwa}^{pmL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmL}}{\partial v^c \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} w_{0p}^a C \left[\left(\frac{1}{b} \right)^2 \frac{L}{2} + \nu \left(\frac{p\pi}{L} \right)^2 \frac{1}{b} \frac{Lb}{6} \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wab}} \sum_{n=1}^{N_{wb}} w_{0pn}^b C \left[\nu \left(\frac{p\pi}{L} \right)^2 \frac{1}{b} I_n^{ys} \frac{L}{2} \right] \dot{w}_p^a
\end{aligned} \tag{96}$$

$$\begin{aligned}
\mathbf{K}_{vcwb}^{pmL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmL}}{\partial v^c \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wab}} \sum_{q=1}^{N_{wb}} w_{0p}^a C \left[\nu \frac{1}{b} \left(\frac{p\pi}{L} \right)^2 I_q^{ys} \frac{L}{2} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} w_{0pq}^b C \left[\frac{1}{b} \left(\frac{q\pi}{b} \right)^2 + \nu \frac{1}{b} \left(\frac{p\pi}{L} \right)^2 \right] \frac{Lb}{4} \dot{w}_{pq}^b
\end{aligned} \tag{97}$$

B.2.6 \mathbf{K}_{ww}^{pmL} -matrix

$$\begin{aligned}
\mathbf{K}_{wawa}^{pmL} \dot{w}^a &= \frac{\partial^2 U^{pmL}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} w_{0m}^a w_{0r}^a C \left[\frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} \frac{b}{5} I_{pfmr}^{cccc} + \frac{1-\nu}{2} \frac{m\pi}{L} \frac{r\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{3} I_{pfmr}^{sscc} \right. \\
&\quad + \left(\frac{1}{b}\right)^3 I_{pfmr}^{ssss} + \frac{1-\nu}{2} \left(\frac{1}{b}\right)^2 \frac{p\pi}{L} \frac{f\pi}{L} \frac{b}{3} I_{mrpf}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{3} I_{mrpf}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{f\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{3} I_{prfm}^{sscc} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} w_{0mn}^b w_{0r}^a C \left[2 \frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_n^{yyys} I_{pfmr}^{cccc} \right. \\
&\quad + (1-\nu) \frac{m\pi}{L} \frac{r\pi}{L} \left(\frac{1}{b}\right)^2 I_n^{ys} I_{pfmr}^{sscc} + (1-\nu) \frac{n\pi}{b} \frac{p\pi}{L} \frac{f\pi}{L} \frac{1}{b} I_n^{yyc} I_{mrpf}^{sscc} \\
&\quad + \frac{1+\nu}{2} \frac{r\pi}{L} \frac{n\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_n^{yyc} I_{fmpr}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 I_n^{ys} I_{frmp}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{r\pi}{L} \frac{n\pi}{b} \frac{f\pi}{L} \frac{1}{b} I_n^{yyc} I_{pmfr}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{f\pi}{L} \left(\frac{1}{b}\right)^2 I_n^{ys} I_{prfm}^{sscc} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0mn}^b w_{0rs}^b C \left[\frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{ns}^{yyss} I_{pfmr}^{cccc} + \frac{1-\nu}{2} \frac{m\pi}{L} \frac{r\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{2} \delta_{n,s} I_{pfmr}^{sscc} \right. \\
&\quad + \frac{n\pi}{b} \frac{s\pi}{b} \left(\frac{1}{b}\right)^2 \frac{b}{2} \delta_{n,s} I_{pfmr}^{ssss} + \frac{1-\nu}{2} \frac{n\pi}{b} \frac{s\pi}{b} \frac{p\pi}{L} \frac{f\pi}{L} I_{ns}^{yycc} I_{mrpf}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{s\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_{sn}^{ycs} I_{frmp}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{s\pi}{b} \frac{f\pi}{L} \frac{1}{b} I_{sn}^{ycs} I_{prfm}^{sscc} \right] \dot{w}_p^a
\end{aligned} \tag{98}$$

$$\begin{aligned}
\mathbf{K}_{wawb}^{pmL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmL}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} w_{0m}^a w_{0r}^a C \left[\frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_q^{yyyy} I_{fpmr}^{cccc} \right. \\
&\quad + \frac{1-\nu}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_q^{yycc} I_{fpmr}^{sscc} + \frac{1-\nu}{2} \left(\frac{1}{b} \right)^2 \frac{f\pi}{L} \frac{p\pi}{L} I_q^{yys} I_{mrfp}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} I_q^{yycc} I_{prfm}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{1}{b} \right)^2 I_q^{yys} I_{frpm}^{sscc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} w_{0mn}^b w_{0r}^a C \left[2 \frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{qn}^{yyss} I_{fpmr}^{cccc} + (1-\nu) \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qn}^{yys} I_{fpmr}^{sscc} \right. \\
&\quad \left. + 2 \left(\frac{1}{b} \right)^2 \frac{n\pi}{b} \frac{q\pi}{b} \frac{b}{2} \delta_{q,n} I_{fpmr}^{ssss} + (1-\nu) \frac{n\pi}{b} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_{nq}^{yys} I_{mrfp}^{sscc} \right. \\
&\quad + \frac{1+\nu}{2} \frac{r\pi}{L} \frac{n\pi}{b} \frac{f\pi}{L} \frac{q\pi}{b} I_{qn}^{yycc} I_{pmfr}^{sscc} + \frac{1+\nu}{2} \frac{r\pi}{L} \frac{n\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_{nq}^{yys} I_{fmp}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qn}^{yys} I_{prfm}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{1}{b} \right)^2 \frac{b}{2} \delta_{q,n} I_{frpm}^{sscc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0mn}^b w_{0rs}^b C \left[\frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{qns}^{yyss} I_{fpmr}^{cccc} + \frac{1-\nu}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qns}^{yys} I_{fpmr}^{sscc} \right. \\
&\quad + \frac{n\pi}{b} \frac{s\pi}{b} \frac{q\pi}{b} \frac{1}{b} I_{qns}^{yys} I_{pfmr}^{ssss} + \frac{1-\nu}{2} \frac{n\pi}{b} \frac{s\pi}{b} \frac{p\pi}{L} \frac{f\pi}{L} I_{qns}^{yys} I_{mrpf}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{s\pi}{b} \frac{f\pi}{L} \frac{q\pi}{b} I_{nqs}^{yys} I_{prfm}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{s\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_{sqn}^{yys} I_{frpm}^{sscc} \right] \dot{w}_{pq}^b
\end{aligned} \tag{99}$$

$$\begin{aligned}
\mathbf{K}_{wbwb}^{pmL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmL}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} w_{0m}^a w_{0r}^a C \left[\frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{yg}^{yyss} I_{fpmr}^{cccc} \right. \\
&\quad + \frac{1-\nu}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{g\pi}{b} I_{yg}^{yycc} I_{fpmr}^{sscc} \\
&\quad + \left(\frac{1}{b}\right)^2 \left(\frac{q\pi}{b}\right)^2 \frac{b}{2} \delta_{q,g} I_{pfmr}^{ssss} + \frac{1-\nu}{2} \left(\frac{1}{b}\right)^2 \frac{p\pi}{L} \frac{f\pi}{L} \frac{b}{2} \delta_{q,g} I_{mrfp}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{p\pi}{L} \frac{g\pi}{b} \frac{1}{b} I_{gq}^{yics} I_{frpm}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{gq}^{yics} I_{prfm}^{sscc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} w_{0mn}^b w_{0r}^a C \left[2 \frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{qgn}^{ysss} I_{fpmr}^{cccc} \right. \\
&\quad + (1-\nu) \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{g\pi}{b} I_{nqg}^{yscc} I_{fpmr}^{sscc} \\
&\quad + 2 \frac{n\pi}{b} \frac{q\pi}{b} \frac{g\pi}{b} \frac{1}{b} I_{qgn}^{iccc} I_{fpmr}^{ssss} + (1-\nu) \frac{n\pi}{b} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_{nqg}^{css} I_{mrfp}^{sscc} \\
&\quad + \frac{1+\nu}{2} \frac{r\pi}{L} \frac{n\pi}{b} \frac{p\pi}{L} \frac{g\pi}{b} I_{qgn}^{yscc} I_{fpmr}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{p\pi}{L} \frac{g\pi}{b} \frac{1}{b} I_{qgn}^{css} I_{frpm}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{r\pi}{L} \frac{n\pi}{b} \frac{f\pi}{L} \frac{q\pi}{b} I_{gqn}^{yscc} I_{pmfr}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qgn}^{css} I_{prfm}^{sscc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0mn}^b w_{0rs}^b C \left[\frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{qgns}^{ssss} I_{fpmr}^{cccc} \right. \\
&\quad + \frac{1-\nu}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{g\pi}{b} I_{pfmr}^{sscc} I_{nsqg}^{sscc} \\
&\quad + \frac{n\pi}{b} \frac{s\pi}{b} \frac{q\pi}{b} \frac{g\pi}{b} I_{pfmr}^{ssss} I_{qgns}^{cccc} + \frac{1-\nu}{2} \frac{n\pi}{b} \frac{s\pi}{b} \frac{p\pi}{L} \frac{f\pi}{L} I_{mrpf}^{sscc} I_{qgns}^{sscc} \\
&\quad \left. + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{s\pi}{b} \frac{p\pi}{L} \frac{g\pi}{b} I_{frpm}^{sscc} I_{qngs}^{sscc} + \frac{1+\nu}{2} \frac{m\pi}{L} \frac{s\pi}{b} \frac{f\pi}{L} \frac{q\pi}{b} I_{prfm}^{sscc} I_{gnqs}^{sscc} \right] \dot{w}_{pq}^b
\end{aligned} \tag{100}$$

B.2.7 \mathbf{K}_{uw}^{pmNL} -matrix

$$\begin{aligned}
\mathbf{K}_{uawa}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial u_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{b}{4} I_{fpm}^{ccc} + \frac{1-\nu}{2} \left(\frac{1}{b}\right)^2 \frac{b}{2} \frac{m\pi}{L} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{2} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{2} I_{pfm}^{css} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_n^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \left(\frac{1}{b}\right)^2 \frac{m\pi}{L} I_n^s I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{1}{b} \frac{n\pi}{b} I_n^{yc} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{p\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_n^{yc} I_{pfm}^{css} \right] \dot{w}_p^a
\end{aligned} \tag{101}$$

$$\begin{aligned}
\mathbf{K}_{uawb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial u_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_q^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{q\pi}{b} \frac{m\pi}{L} \frac{1}{b} I_q^{yc} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_q^{yc} I_{fpm}^{css} + \frac{1-\nu}{2} \left(\frac{1}{b}\right)^2 \frac{p\pi}{L} I_q^s I_{pfm}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_{qn}^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{1}{b} \frac{q\pi}{b} \frac{m\pi}{L} I_{qn}^s I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{q\pi}{b} \frac{n\pi}{b} I_{qn}^{ycc} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{p\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_{nq}^{cs} I_{pfm}^{css} \right] \dot{w}_{pq}^b
\end{aligned} \tag{102}$$

$$\begin{aligned}
\mathbf{K}_{ubwa}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial u_{fg}^b \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_g^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{m\pi}{L} \frac{1}{b} I_g^{yc} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \left(\frac{1}{b}\right)^2 I_g^s I_{fpm}^{css} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_g^{yc} I_{pfm}^{css} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_{gn}^{yys} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{1}{b} \frac{g\pi}{b} \frac{m\pi}{L} I_{gn}^{cs} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_{ng}^{cs} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} \frac{n\pi}{b} I_{gn}^{ycc} I_{pfm}^{css} \right] \dot{w}_p^a
\end{aligned} \tag{103}$$

$$\begin{aligned}
\mathbf{K}_{ubwb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial u_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_{gq}^{yss} I_{fpm}^{ccc} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{q\pi}{b} \frac{m\pi}{L} I_{gq}^{ycc} I_{mfp}^{css} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qg}^{ics} I_{fpm}^{css} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_{gq}^{ics} I_{pfm}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b C \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} I_{fpm}^{ccc} I_{gqn}^{sss} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{q\pi}{b} \frac{m\pi}{L} I_{mfp}^{css} I_{ngq}^{scc} \right. \\
&\quad \left. + \nu \frac{f\pi}{L} \frac{q\pi}{b} \frac{n\pi}{b} I_{fpm}^{css} I_{gqn}^{scc} + \frac{1-\nu}{2} \frac{g\pi}{b} \frac{p\pi}{L} \frac{n\pi}{b} I_{qgn}^{scc} I_{pfm}^{css} \right] \dot{w}_{pq}^b
\end{aligned} \tag{104}$$

$$\begin{aligned}
\mathbf{K}_{ucwa}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial u^c \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} w_p^a C \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 \frac{Lb}{6} + \nu \frac{1}{L} \frac{1}{b} \frac{L}{2} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wab}} \sum_{n=1}^{N_{wb}} w_{pn}^b C \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 I_n^{ys} \frac{L}{2} \right] \dot{w}_p^a
\end{aligned} \tag{105}$$

$$\begin{aligned}
\mathbf{K}_{ucwb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial u^c \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wab}} \sum_{q=1}^{N_{wb}} w_p^a C \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 I_q^{ys} \frac{L}{2} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} w_{pq}^b C \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 + \nu \frac{1}{L} \left(\frac{q\pi}{b} \right)^2 \right] \frac{Lb}{4} \dot{w}_{pq}^b
\end{aligned} \tag{106}$$

B.2.8 \mathbf{K}_{uww}^{pmNL} -matrix

$$\begin{aligned}
\mathbf{K}_{uww}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{i=1}^{M_{ua}} u_i^a C \left[\frac{i\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} \frac{b}{4} I_{ipf}^{ccc} + \nu \frac{i\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{2} I_{ipf}^{css} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{2} I_{pif}^{css} + \frac{1-\nu}{2} \frac{f\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{2} I_{fip}^{css} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{i=1}^{M_{ub}} \sum_{j=1}^{N_{ub}} u_{ij}^b C \left[\frac{i\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_j^{yys} I_{ipf}^{ccc} + \nu \frac{i\pi}{L} \left(\frac{1}{b}\right)^2 I_j^{css} I_{ipf}^{css} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{j\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_j^{yc} I_{pif}^{css} + \frac{1-\nu}{2} \frac{j\pi}{b} \frac{f\pi}{L} \frac{1}{b} I_j^{yc} I_{fip}^{css} \right] \dot{w}_p^a \\
&\quad + u^c C \left[\frac{1}{L} \left(\frac{f\pi}{L}\right)^2 \frac{Lb}{6} + \nu \frac{1}{L} \frac{1}{b} \frac{L}{2} \right] \dot{w}_f^a
\end{aligned} \tag{107}$$

$$\begin{aligned}
\mathbf{K}_{uww}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{ua}} u_i^a C \left[\frac{i\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_q^{yys} I_{ipf}^{ccc} + \nu \frac{i\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_q^{yc} I_{ipf}^{css} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_q^{yc} I_{fip}^{css} + \frac{1-\nu}{2} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 I_q^{cs} I_{pif}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{ub}} \sum_{j=1}^{N_{ub}} u_{ij}^b C \left[\frac{i\pi}{L} \frac{f\pi}{L} \frac{p\pi}{L} I_{jq}^{yss} I_{ifp}^{ccc} + \nu \frac{i\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{jq}^{cs} I_{ifp}^{css} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{j\pi}{b} \frac{f\pi}{L} \frac{q\pi}{b} I_{jq}^{ycc} I_{fip}^{css} + \frac{1-\nu}{2} \frac{j\pi}{b} \frac{p\pi}{L} \frac{1}{b} I_{jq}^{cs} I_{pif}^{css} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{q=1}^{N_{wb}} u^c C \left[\frac{1}{L} \left(\frac{f\pi}{L}\right)^2 I_q^{ys} \frac{L}{2} \right] \dot{w}_{fq}^b
\end{aligned} \tag{108}$$

$$\begin{aligned}
\mathbf{K}_{uwbwb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{ua}} u_i^a C \left[\frac{i\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{qg}^{yss} I_{ipf}^{ccc} + \nu \frac{i\pi}{L} \frac{q\pi}{b} \frac{g\pi}{b} I_{qg}^{ycc} I_{ipf}^{css} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{p\pi}{L} \frac{g\pi}{b} \frac{1}{b} I_{gq}^{cs} I_{pif}^{css} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qg}^{cs} I_{fip}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{ub}} \sum_{j=1}^{N_{ub}} u_{ij}^b C \left[\frac{i\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{ipf}^{icc} I_{jqg}^{sss} + \nu \frac{i\pi}{L} \frac{q\pi}{b} \frac{g\pi}{b} I_{ipf}^{css} I_{jqg}^{scc} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{j\pi}{b} \frac{p\pi}{L} \frac{g\pi}{b} I_{pif}^{css} I_{qjg}^{scc} + \frac{1-\nu}{2} \frac{j\pi}{b} \frac{f\pi}{L} \frac{q\pi}{b} I_{fip}^{css} I_{gjq}^{scc} \right] \dot{w}_{pq}^b \\
&\quad + u^c C \left[\left(\frac{f\pi}{L} \right)^2 + \nu \left(\frac{g\pi}{b} \right)^2 \right] \frac{1}{L} \frac{Lb}{4} \dot{w}_{fg}^b
\end{aligned} \tag{109}$$

B.2.9 \mathbf{K}_{vw}^{pmNL} -matrix

$$\begin{aligned}
\mathbf{K}_{vawa}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial v_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a C \left[\left(\frac{1}{b} \right)^2 I_{fpm}^{css} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} \frac{1}{3} I_{pfm}^{css} \right. \\
&\quad \left. + \nu \frac{p\pi}{L} \frac{m\pi}{L} \frac{1}{b} \frac{1}{3} I_{fpm}^{ccc} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} \frac{1}{3} I_{mfp}^{css} \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b C \left[\nu \frac{p\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_n^{ys} I_{fpm}^{ccc} \right. \\
&\quad \left. - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} I_n^{yyc} I_{pfm}^{css} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_n^{ys} I_{mfp}^{css} \right] \dot{w}_p^a
\end{aligned} \tag{110}$$

$$\begin{aligned}
\mathbf{K}_{vawb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial v_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a C \left[\nu \frac{p\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_q^{ys} I_{fpm}^{ccc} \right. \\
&\quad \left. - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_q^{ys} I_{pfm}^{css} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{m\pi}{L} I_q^{yyc} I_{mfp}^{css} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b C \left[\frac{q\pi}{b} \frac{n\pi}{b} \frac{1}{b} \frac{1}{2} \delta_{q,n} I_{fpm}^{css} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} I_{nq}^{ycc} I_{pfm}^{css} \right. \\
&\quad \left. + \nu \frac{p\pi}{L} \frac{m\pi}{L} \frac{1}{b} \frac{1}{2} \delta_{q,n} I_{fpm}^{ccc} - \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{m\pi}{L} I_{qn}^{ycc} I_{mfp}^{css} \right] \dot{w}_{pq}^b
\end{aligned} \tag{111}$$

$$\begin{aligned}
\mathbf{K}_{vbwa}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial v_{fg}^b \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a C \left[\frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_g^{ys} I_{mfp}^{scc} \right. \\
&\quad \left. + \nu \frac{g\pi}{b} \frac{p\pi}{L} \frac{m\pi}{L} I_g^{yyc} I_{fpm}^{scc} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_g^{ys} I_{pfm}^{scc} \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b C \left[\frac{g\pi}{b} \frac{n\pi}{b} \frac{1}{b} \frac{b}{2} \delta_{g,n} I_{fpm}^{sss} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} I_{ng}^{ycc} I_{mfp}^{scc} \right. \\
&\quad \left. + \nu \frac{g\pi}{b} \frac{p\pi}{L} \frac{m\pi}{L} I_{gn}^{ycc} I_{fpm}^{scc} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} \frac{b}{2} \delta_{g,n} I_{pfm}^{scc} \right] \dot{w}_p^a
\end{aligned} \tag{112}$$

$$\begin{aligned}
\mathbf{K}_{vbwb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial v_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a C \left[\frac{g\pi}{b} \frac{q\pi}{b} \frac{1}{b} \frac{b}{2} \delta_{g,n} I_{fpm}^{sss} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} \frac{b}{2} \delta_{g,n} I_{mfp}^{scc} \right. \\
&\quad \left. + \nu \frac{g\pi}{b} \frac{p\pi}{L} \frac{m\pi}{L} I_{gq}^{ycc} I_{fpm}^{scc} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{m\pi}{L} I_{qg}^{ycc} I_{pfm}^{scc} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b C \left[\frac{g\pi}{b} \frac{q\pi}{b} \frac{n\pi}{b} I_{fpm}^{sss} I_{gqn}^{ccc} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} I_{mfp}^{scc} I_{ngq}^{css} \right. \\
&\quad \left. + \nu \frac{g\pi}{b} \frac{p\pi}{L} \frac{m\pi}{L} I_{fpm}^{scc} I_{gqn}^{css} + \frac{1-\nu}{2} \frac{f\pi}{L} \frac{q\pi}{b} \frac{m\pi}{L} I_{pfm}^{scc} I_{qgn}^{css} \right] \dot{w}_{pq}^b
\end{aligned} \tag{113}$$

$$\begin{aligned}
\mathbf{K}_{vcwa}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial v^c \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} w_p^a C \left[\left(\frac{1}{b} \right)^2 \frac{L}{2} + \nu \left(\frac{p\pi}{L} \right)^2 \frac{1}{b} \frac{Lb}{6} \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wab}} \sum_{n=1}^{N_{wb}} w_{pn}^b C \left[\nu \left(\frac{p\pi}{L} \right)^2 \frac{1}{b} I_n^{ys} \frac{L}{2} \right] \dot{w}_p^a
\end{aligned} \tag{114}$$

$$\begin{aligned}
\mathbf{K}_{vcwb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial v^c \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wab}} \sum_{q=1}^{N_{wb}} w_p^a C \left[\nu \frac{1}{b} \left(\frac{p\pi}{L} \right)^2 I_q^{ys} \frac{L}{2} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} w_{pq}^b C \left[\frac{1}{b} \left(\frac{q\pi}{b} \right)^2 + \nu \frac{1}{b} \left(\frac{p\pi}{L} \right)^2 \right] \frac{Lb}{4} \dot{w}_{pq}^b
\end{aligned} \tag{115}$$

B.2.10 \mathbf{K}_{vw}^{pmNL} -matrix

$$\begin{aligned}
\mathbf{K}_{vwawa}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{i=1}^{M_{va}} v_i^a C \left[\left(\frac{1}{b} \right)^2 I_{ipf}^{css} + \nu \frac{p\pi}{L} \frac{f\pi}{L} \frac{1}{b} \frac{1}{3} I_{ipf}^{ccc} \right. \\
&\quad \left. - \frac{1-\nu}{2} \frac{i\pi}{L} \frac{p\pi}{L} \frac{1}{b} \frac{1}{3} I_{pif}^{css} - \frac{1-\nu}{2} \frac{i\pi}{L} \frac{f\pi}{L} \frac{1}{b} \frac{1}{3} I_{fip}^{css} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{i=1}^{M_{vb}} \sum_{j=1}^{N_{vb}} v_{ij}^b C \left[\nu \frac{j\pi}{b} \frac{p\pi}{L} \frac{f\pi}{L} I_j^{yyc} I_{ipf}^{scc} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{i\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_j^{ys} I_{fip}^{scc} + \frac{1-\nu}{2} \frac{i\pi}{L} \frac{f\pi}{L} \frac{1}{b} I_j^{ys} I_{pif}^{scc} \right] \dot{w}_p^a \\
&\quad + v^c C \left[\left(\frac{1}{b} \right)^2 \frac{L}{2} + \nu \left(\frac{f\pi}{L} \right)^2 \frac{1}{b} \frac{Lb}{6} \right] \dot{w}_f^a
\end{aligned} \tag{116}$$

$$\begin{aligned}
\mathbf{K}_{vwawb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{va}} v_i^a C \left[\nu \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_q^{ys} I_{ifp}^{ccc} \right. \\
&\quad \left. - \frac{1-\nu}{2} \frac{i\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} I_q^{yyc} I_{fip}^{css} - \frac{1-\nu}{2} \frac{i\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_q^{ys} I_{pif}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{vb}} \sum_{j=1}^{N_{vb}} v_{ij}^b C \left[\frac{j\pi}{b} \frac{q\pi}{b} \frac{1}{b} \frac{1}{2} \delta_{j,q} I_{ifp}^{sss} + \nu \frac{j\pi}{b} \frac{f\pi}{L} \frac{p\pi}{L} I_{jq}^{yyc} I_{ifp}^{scc} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{i\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} I_{qj}^{ycs} I_{pif}^{scc} + \frac{1-\nu}{2} \frac{i\pi}{L} \frac{p\pi}{L} \frac{1}{b} \frac{1}{2} \delta_{j,q} I_{fip}^{scc} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{q=1}^{N_{wb}} v^c C \left[\nu \left(\frac{f\pi}{L} \right)^2 \frac{1}{b} \frac{L}{2} I_q^{ys} \right] \dot{w}_{fq}^b
\end{aligned} \tag{117}$$

$$\begin{aligned}
\mathbf{K}_{vbwbw}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{va}} v_i^a C \left[\frac{q\pi}{b} \frac{g\pi}{b} \frac{1}{b} \frac{b}{2} \delta_{q,g} I_{ipf}^{css} + \nu \frac{p\pi}{L} \frac{f\pi}{L} \frac{1}{b} \frac{b}{2} \delta_{q,g} I_{ipf}^{ccc} \right. \\
&\quad \left. - \frac{1-\nu}{2} \frac{i\pi}{L} \frac{p\pi}{L} \frac{g\pi}{b} I_{gq}^{ycs} I_{pij}^{css} - \frac{1-\nu}{2} \frac{i\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} I_{gq}^{ycs} I_{fip}^{css} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{vb}} v_i^b C \left[\frac{j\pi}{b} \frac{q\pi}{b} \frac{g\pi}{b} I_{ipf}^{sss} I_{jqg}^{ccc} + \nu \frac{j\pi}{b} \frac{p\pi}{L} \frac{f\pi}{L} I_{ipf}^{scc} I_{jqg}^{css} \right. \\
&\quad \left. + \frac{1-\nu}{2} \frac{i\pi}{L} \frac{p\pi}{L} \frac{g\pi}{b} I_{fip}^{scc} I_{gjq}^{css} + \frac{1-\nu}{2} \frac{i\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} I_{pij}^{scc} I_{qjg}^{css} \right] \dot{w}_{pq}^b \\
&\quad + v^c C \left[\left(\frac{g\pi}{b} \right)^2 + \nu \left(\frac{f\pi}{L} \right)^2 \right] \frac{1}{b} \frac{Lb}{4} \dot{w}_{fg}^b
\end{aligned} \tag{118}$$

B.2.11 $\mathbf{K}_{wawaw}^{pmNL}$ -matrix

$$\begin{aligned}
\mathbf{K}_{wawaw}^{pmNL} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{pmNL}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} (w_m^a w_r^a + 2w_m^a w_{0r}^a) C \left[\frac{3}{10} \frac{p\pi}{L} \frac{f\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} b I_{pfmr}^{cccc} + \frac{3}{2} \left(\frac{1}{b} \right)^3 I_{pfmr}^{ssss} \right. \\
&\quad + \frac{1}{6} \frac{f\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_{mrfp}^{sscc} + \frac{1}{6} \frac{f\pi}{L} \frac{m\pi}{L} \frac{1}{b} I_{prfm}^{sscc} + \frac{1}{6} \frac{f\pi}{L} \frac{r\pi}{L} \frac{1}{b} I_{mpfr}^{sscc} \\
&\quad \left. + \frac{1}{6} \frac{m\pi}{L} \frac{r\pi}{L} \frac{1}{b} I_{fpmr}^{sscc} + \frac{1}{6} \frac{m\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_{frmp}^{sscc} + \frac{1}{6} \frac{r\pi}{L} \frac{p\pi}{L} \frac{1}{b} I_{fmpfr}^{sscc} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} (w_m^a w_{rs}^b + w_{0m}^a w_{rs}^b + w_m^a w_{0rs}^b) C \left[3 \frac{p\pi}{L} \frac{f\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} I_s^{yyys} I_{pfmr}^{cccc} \right. \\
&\quad + \frac{p\pi}{L} \frac{f\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_s^{yycc} I_{mrfp}^{sscc} + \frac{m\pi}{L} \frac{f\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_s^{yycc} I_{prfm}^{sscc} + \frac{m\pi}{L} \frac{p\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_s^{yycc} I_{fmpfr}^{sscc} \\
&\quad + \frac{p\pi}{L} \frac{r\pi}{L} \left(\frac{1}{b} \right)^2 I_s^{ys} I_{fmpfr}^{sscc} + \frac{m\pi}{L} \frac{r\pi}{L} \left(\frac{1}{b} \right)^2 I_s^{ys} I_{fpmr}^{sscc} + \frac{f\pi}{L} \frac{r\pi}{L} \left(\frac{1}{b} \right)^2 I_s^{ys} I_{mpfr}^{sscc} \left. \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} (w_{mn}^b w_{rs}^b + 2w_{mn}^b w_{0rs}^b) C \left(\frac{3}{2} \frac{p\pi}{L} \frac{f\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} I_{ns}^{yyss} I_{pfmr}^{cccc} + \frac{3}{4} \frac{n\pi}{b} \frac{s\pi}{b} \frac{1}{b} \delta_{n,s} I_{frmp}^{ssss} \right. \\
&\quad + \frac{1}{2} \frac{p\pi}{L} \frac{f\pi}{L} \frac{n\pi}{b} \frac{s\pi}{b} I_{ns}^{yycc} I_{mrfp}^{sscc} + \frac{1}{4} \frac{m\pi}{L} \frac{r\pi}{L} \frac{1}{b} \delta_{n,s} I_{fpmr}^{sscc} + \frac{1}{2} \frac{p\pi}{L} \frac{m\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_{sn}^{ycs} I_{fmpfr}^{sscc} \\
&\quad \left. + \frac{1}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_{sn}^{ycs} I_{prfm}^{sscc} + \frac{1}{2} \frac{f\pi}{L} \frac{r\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_{ns}^{ycs} I_{mpfr}^{sscc} + \frac{1}{2} \frac{p\pi}{L} \frac{r\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_{ns}^{ycs} I_{fmpfr}^{sscc} \right) \dot{w}_p^a
\end{aligned} \tag{119}$$

$$\begin{aligned}
\mathbf{K}_{wawb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} (w_m^a w_r^a + 2w_m^a w_{0r}^a) C \left[\frac{3}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} I_q^{yyyy} I_{fpmr}^{cccc} \right. \\
&\quad + \frac{1}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_q^{yyyc} I_{fpmr}^{sscc} + \frac{1}{2} \frac{f\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_q^{yyyc} I_{mpfr}^{sscc} + \frac{1}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_q^{yyyc} I_{prfm}^{sscc} \\
&\quad \left. + \frac{1}{2} \frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 I_q^{yys} I_{mrfp}^{sscc} + \frac{1}{2} \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 I_q^{yys} I_{frmp}^{sscc} + \frac{1}{2} \frac{r\pi}{L} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 I_q^{yys} I_{fmpr}^{sscc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} (w_m^a w_{rs}^b + w_{0m}^a w_{rs}^b + w_m^a w_{0rs}^b) C \left[3 \frac{f\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} I_{sq}^{yyss} I_{fmrp}^{cccc} \right. \\
&\quad + 3 \frac{s\pi}{b} \frac{q\pi}{b} \left(\frac{1}{b}\right)^2 \frac{b}{2} \delta_{s,q} I_{fmrp}^{ssss} + \frac{f\pi}{L} \frac{m\pi}{L} \frac{s\pi}{b} \frac{q\pi}{b} I_{sq}^{yycc} I_{prfm}^{sscc} + \frac{f\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qs}^{yys} I_{mpfr}^{sscc} \\
&\quad + \frac{f\pi}{L} \frac{p\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_{sq}^{yys} I_{mrfp}^{sscc} + \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qs}^{yys} I_{fpmr}^{sscc} + \frac{m\pi}{L} \frac{p\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_{sq}^{yys} I_{frmp}^{sscc} \\
&\quad \left. + \frac{r\pi}{L} \frac{p\pi}{L} \left(\frac{1}{b}\right)^2 \frac{b}{2} \delta_{s,q} I_{fmrp}^{sscc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} (w_{mn}^b w_{rs}^b + 2w_{mn}^b w_{0rs}^b) C \left[\frac{3}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} I_{qns}^{ysss} I_{fmpr}^{cccc} \right. \\
&\quad + \frac{3}{2} \frac{q\pi}{b} \frac{n\pi}{b} \frac{s\pi}{b} \frac{1}{b} I_{qns}^{cccc} I_{fpmr}^{ssss} + \frac{1}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} \frac{s\pi}{b} I_{qns}^{yssc} I_{mrfp}^{sscc} + \frac{1}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{q\pi}{b} \frac{s\pi}{b} I_{nqs}^{yssc} I_{prfm}^{sscc} \\
&\quad + \frac{1}{2} \frac{f\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{n\pi}{b} I_{sqn}^{yssc} I_{mpfr}^{sscc} + \frac{1}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qns}^{css} I_{fpmr}^{sscc} \\
&\quad \left. + \frac{1}{2} \frac{p\pi}{L} \frac{r\pi}{L} \frac{n\pi}{b} \frac{1}{b} I_{nqs}^{css} I_{fmpr}^{sscc} + \frac{1}{2} \frac{m\pi}{L} \frac{p\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_{sqn}^{css} I_{frmp}^{sscc} \right] \dot{w}_{pq}^b
\end{aligned} \tag{120}$$

$$\begin{aligned}
\mathbf{K}_{wbwb}^{pmNL} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{pmNL}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} (w_{mn}^b w_{rs}^b + 2w_{mn}^b w_{0rs}^b) C \left[\frac{3}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} I_{fpmr}^{cccc} I_{gqns}^{ssss} \right. \\
&\quad + \frac{3}{2} \frac{g\pi}{b} \frac{q\pi}{b} \frac{n\pi}{b} \frac{s\pi}{b} I_{fpmr}^{ssss} I_{gqns}^{cccc} + \frac{1}{2} \frac{f\pi}{L} \frac{p\pi}{L} \frac{n\pi}{b} \frac{s\pi}{b} I_{mrfp}^{sscc} I_{gqns}^{sscc} + \frac{1}{2} \frac{f\pi}{L} \frac{m\pi}{L} \frac{q\pi}{b} \frac{s\pi}{b} I_{prfm}^{sscc} I_{gnqs}^{sscc} \\
&\quad + \frac{1}{2} \frac{f\pi}{L} \frac{r\pi}{L} \frac{n\pi}{b} \frac{q\pi}{b} I_{mpfr}^{sscc} I_{gsnq}^{sscc} + \frac{1}{2} \frac{m\pi}{L} \frac{p\pi}{L} \frac{g\pi}{b} \frac{s\pi}{b} I_{frmp}^{sscc} I_{nqgs}^{sscc} + \frac{1}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{g\pi}{b} I_{fpmr}^{sscc} I_{nsqg}^{sscc} \\
&\quad \left. + \frac{1}{2} \frac{p\pi}{L} \frac{r\pi}{L} \frac{g\pi}{b} \frac{n\pi}{b} I_{fmpr}^{sscc} I_{qsgn}^{sscc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} (w_m^a w_{rs}^b + w_{0m}^a w_{rs}^b + w_m^a w_{0rs}^b) C \left[3 \frac{m\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} \frac{r\pi}{L} I_{yqgs}^{ysss} I_{mpfr}^{cccc} \right. \\
&\quad + 3 \frac{q\pi}{b} \frac{g\pi}{b} \frac{s\pi}{b} \frac{1}{b} I_{qgs}^{cccc} I_{fmrp}^{ssss} + \frac{m\pi}{L} \frac{p\pi}{L} \frac{g\pi}{b} \frac{s\pi}{b} I_{yqgs}^{sscc} I_{fmrp}^{sscc} + \frac{m\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} \frac{s\pi}{b} I_{yqgs}^{sscc} I_{prfm}^{sscc} \\
&\quad + \frac{m\pi}{L} \frac{r\pi}{L} \frac{g\pi}{b} \frac{q\pi}{b} I_{sgq}^{sscc} I_{fpmr}^{sscc} + \frac{f\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{qgs}^{css} I_{mpfr}^{sscc} + \frac{p\pi}{L} \frac{r\pi}{L} \frac{g\pi}{b} \frac{1}{b} I_{qgs}^{css} I_{fmpr}^{sscc} \\
&\quad \left. + \frac{p\pi}{L} \frac{f\pi}{L} \frac{s\pi}{b} \frac{1}{b} I_{sgq}^{css} I_{mrpf}^{sscc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} \left[(w_m^a w_r^a + 2w_m^a w_{0r}^a) C \left(\frac{3}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{p\pi}{L} \frac{f\pi}{L} I_{yqg}^{yyss} I_{mrpf}^{cccc} + \frac{3}{4} \frac{q\pi}{b} \frac{g\pi}{b} \frac{1}{b} \delta_{q,g} I_{mrpf}^{ssss} \right. \right. \\
&\quad + \frac{1}{2} \frac{p\pi}{L} \frac{f\pi}{L} \left(\frac{1}{b} \right)^2 \frac{b}{2} \delta_{q,g} I_{mrfp}^{sscc} + \frac{1}{2} \frac{m\pi}{L} \frac{r\pi}{L} \frac{q\pi}{b} \frac{g\pi}{b} I_{yqg}^{yycc} I_{fpmr}^{sscc} + \frac{1}{2} \frac{m\pi}{L} \frac{p\pi}{L} \frac{g\pi}{b} \frac{1}{b} I_{yqg}^{yics} I_{fmrp}^{sscc} \\
&\quad \left. \left. + \frac{1}{2} \frac{m\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{yqg}^{yics} I_{prfm}^{sscc} + \frac{1}{2} \frac{r\pi}{L} \frac{f\pi}{L} \frac{q\pi}{b} \frac{1}{b} I_{yqg}^{yics} I_{mpfr}^{sscc} + \frac{1}{2} \frac{r\pi}{L} \frac{p\pi}{L} \frac{g\pi}{b} \frac{1}{b} I_{yqg}^{yics} I_{fmpr}^{sscc} \right) \dot{w}_{pq}^b \right] \\
\end{aligned} \tag{121}$$

B.3 The load vector $-\dot{\Lambda}$ G-vector for the plate

The only nonzero contribution for the load vector is

$$\mathbf{G}_{uc} \dot{\Lambda} = S_{0x} b t \dot{\Lambda} \tag{122}$$

For the load considered, all the other elements in the \mathbf{G} -vector are zero:

$$\begin{aligned}
\mathbf{G}_{ua}^{pmL} &= 0, & \mathbf{G}_{ub}^{pmL} &= 0, & \mathbf{G}_{va}^{pmL} &= 0, & \mathbf{G}_{vb}^{pmL} &= 0, \\
\mathbf{G}_{vc}^{pmL} &= 0, & \mathbf{G}_{wa}^{pmL} &= 0, & \mathbf{G}_{wb}^{pmL} &= 0
\end{aligned} \tag{123}$$

B.4 \mathbf{K}^G -matrices

B.4.1 \mathbf{K}_{ww}^G -matrix

The geometrical stiffness matrix \mathbf{K}_{ww}^G is calculated by differentiation with respect to the out-of-plane displacements of the potential of the external loads without imperfection.

$$\mathbf{K}_{wawa}^G \dot{\mathbf{w}}^a = \frac{\partial^2 T(w_0 = 0)}{\partial w_f^a \partial w_p^a} \dot{w}_p^a = -\Lambda \frac{S_{x0} L b t}{6} \left(\frac{f\pi}{L} \right)^2 \dot{w}_f^a \quad (124)$$

$$\mathbf{K}_{wawb}^G \dot{\mathbf{w}}^b = \frac{\partial^2 T(w_0 = 0)}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b = \sum_{q=1}^{N_{wb}} -\Lambda \frac{S_{x0} L t}{2} \left(\frac{f\pi}{L} \right)^2 I_q^{ys} \dot{w}_{fq}^b \quad (125)$$

$$\mathbf{K}_{wbwa}^G \dot{\mathbf{w}}^b = \frac{\partial^2 T(w_0 = 0)}{\partial w_{fg}^b \partial w_p^a} \dot{w}_p^a = -\Lambda \frac{S_{x0} L t}{2} \left(\frac{f\pi}{L} \right)^2 I_g^{ys} \dot{w}_f^a \quad (126)$$

$$\mathbf{K}_{wbwb}^G \dot{\mathbf{w}}^b = \frac{\partial^2 T(w_0 = 0)}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b = -\Lambda \frac{S_{x0} L b t}{4} \left(\frac{f\pi}{L} \right)^2 \dot{w}_{fg}^b \quad (127)$$

C Rate form of energy contributions of a stiffener

C.1 $\mathbf{K}^{s,x}$ -matrices for stiffeners in x -direction

C.1.1 $\mathbf{K}_{uu}^{sL,x}$ -matrix

$$\mathbf{K}_{uaua}^{sL,x} \dot{\mathbf{u}}^a = \frac{\partial^2 U^{sL,x}}{\partial u_f^a \partial u_p^a} \dot{u}_p^a = E A_s \left[\left(\frac{f\pi}{L} \right)^2 \left(\frac{y_s}{b} \right)^2 \frac{L}{2} \right] \dot{u}_f^a \quad (128)$$

$$\mathbf{K}_{ubub}^{sL,x} \dot{\mathbf{u}}^b = \frac{\partial^2 U^{sL,x}}{\partial u_{fg}^b \partial u_{pq}^b} \dot{u}_{pq}^b = \sum_{q=1}^{N_{ub}} E A_s \left[\left(\frac{f\pi}{L} \right)^2 \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \frac{L}{2} \right] \dot{u}_{fq}^b \quad (129)$$

$$\mathbf{K}_{ucuc}^{sL,x} \dot{\mathbf{u}}^c = \frac{\partial^2 U^{sL,x}}{\partial u^c \partial u^c} \dot{u}^c = E A_s \frac{1}{L} \dot{u}^c \quad (130)$$

$$\mathbf{K}_{uaub}^{sL,x} \dot{\mathbf{u}}^b = \frac{\partial^2 U^{sL,x}}{\partial u_f^a \partial u_{pq}^b} \dot{u}_{pq}^b = \sum_{q=1}^{N_{ub}} E A_s \left[\left(\frac{f\pi}{L} \right)^2 \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \frac{L}{2} \right] \dot{u}_{fq}^b \quad (131)$$

$$\mathbf{K}_{ubua}^{sL,x} \dot{\mathbf{u}}^a = \frac{\partial^2 U^{sL,x}}{\partial u_f^b \partial u_p^a} \dot{u}_p^a = EA_s \left[\left(\frac{f\pi}{L} \right)^2 \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \frac{L}{2} \right] \dot{u}_f^a \quad (132)$$

$$\mathbf{K}_{uauc}^{sL,x} = 0, \quad \mathbf{K}_{ucua}^{sL,x} = 0, \quad \mathbf{K}_{ubuc}^{sL,x} = 0, \quad \mathbf{K}_{ucub}^{sL,x} = 0 \quad (133)$$

C.1.2 $\mathbf{K}_{uw}^{sL,x}$ -matrix

$$\begin{aligned} \mathbf{K}_{uawa}^{sL,x} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sL,x}}{\partial u_f^a \partial w_p^a} \dot{w}_p^a \\ &= \sum_{p=1}^{M_{wa}} e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \left(\frac{y_s}{b} \right)^2 I_{fcp}^{cs} \right] \dot{w}_p^a \\ &\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0m}^a EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \left(\frac{y_s}{b} \right)^3 I_{fpm}^{ccc} \right] \dot{w}_p^a \\ &\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{n\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_p^a \end{aligned} \quad (134)$$

$$\begin{aligned} \mathbf{K}_{uawb}^{sL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,x}}{\partial u_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\ &= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) I_{fcp}^{cs} \right] \dot{w}_{pq}^b \\ &\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_{pq}^b \\ &\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_{pq}^b \end{aligned} \quad (135)$$

$$\begin{aligned} \mathbf{K}_{ubwa}^{sL,x} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sL,x}}{\partial u_f^b \partial w_p^a} \dot{w}_p^a \\ &= \sum_{p=1}^{M_{wa}} e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) I_{fcp}^{cs} \right] \dot{w}_p^a \\ &\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0m}^a EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{g\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_p^a \\ &\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_p^a \end{aligned} \quad (136)$$

$$\begin{aligned}
\mathbf{K}_{ubwb}^{sL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,x}}{\partial u_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{fp}^{cs} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_{pq}^b
\end{aligned} \tag{137}$$

$$\begin{aligned}
\mathbf{K}_{ucwa}^{sL,x} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sL,x}}{\partial u^c \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} e_c EA_s \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 \frac{y_s}{b} I_p^s \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wa}} w_{0p}^a EA_s \left[\left(\frac{p\pi}{L} \right)^2 \left(\frac{y_s}{b} \right)^2 \frac{1}{2} \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wawb}} \sum_{n=1}^{N_{wb}} w_{0pn}^b EA_s \left[\left(\frac{p\pi}{L} \right)^2 \frac{y_s}{b} \sin\left(\frac{n\pi}{b} y_s\right) \frac{1}{2} \right] \dot{w}_p^a
\end{aligned} \tag{138}$$

$$\begin{aligned}
\mathbf{K}_{ucwb}^{sL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,x}}{\partial u^c \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} e_c EA_s \left[\frac{1}{L} \left(\frac{p\pi}{L} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) I_p^s \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wawb}} \sum_{q=1}^{N_{wb}} w_{0p}^a EA_s \left[\left(\frac{p\pi}{L} \right)^2 \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \frac{1}{2} \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wawb}} \sum_{q=1}^{N_{wb}} \sum_{n=1}^{N_{wb}} w_{0pn}^b EA_s \left[\left(\frac{p\pi}{L} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \frac{1}{2} \right] \dot{w}_{pq}^b
\end{aligned} \tag{139}$$

C.1.3 $\mathbf{K}_{ww}^{sL,x}$ -matrix

$$\begin{aligned}
\mathbf{K}_{wawa}^{sL,x} \dot{w}^a &= \frac{\partial^2 U^{sL,x}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= EI \left(\frac{f\pi}{L} \right)^4 \left(\frac{y_s}{b} \right)^2 \frac{L}{2} \dot{w}_f^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0m}^a e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \frac{m\pi}{L} I_{pfm}^{scc} \right. \\
&\quad \left. + \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 \frac{m\pi}{L} I_{fpm}^{scc} \right] \left(\frac{y_s}{b} \right)^3 \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \frac{m\pi}{L} I_{pfm}^{scc} \right. \\
&\quad \left. + \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 \frac{m\pi}{L} I_{fpm}^{scc} \right] \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{n\pi}{b} y_s\right) \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} w_{0m}^a w_{0r}^a EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \left(\frac{y_s}{b} \right)^4 I_{pfmr}^{cccc} \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0m}^a w_{0rs}^b 2EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \left(\frac{y_s}{b} \right)^3 \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0mn}^b w_{0rs}^b EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{n\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \dot{w}_p^a
\end{aligned} \tag{140}$$

$$\begin{aligned}
\mathbf{K}_{wawb}^{sL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,x}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{q=1}^{N_{wb}} EI \left(\frac{f\pi}{L} \right)^4 \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \frac{L}{2} \dot{w}_{fq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \frac{m\pi}{L} I_{pfm}^{scc} \right. \\
&\quad \left. + \left(\frac{f\pi}{L} \right)^2 \frac{p\pi}{L} \frac{m\pi}{L} I_{fpm}^{scc} \right] \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \frac{m\pi}{L} I_{pfm}^{scc} \right. \\
&\quad \left. + \left(\frac{f\pi}{L} \right)^2 \frac{p\pi}{L} \frac{m\pi}{L} I_{fpm}^{scc} \right] \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} w_{0m}^a w_{0r}^a EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \left(\frac{y_s}{b} \right)^3 \sin\left(\frac{q\pi}{b} y_s\right) I_{pfmr}^{cccc} \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0m}^a w_{0rs}^b 2EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0mn}^b w_{0rs}^b EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \dot{w}_{pq}^b
\end{aligned} \tag{141}$$

$$\begin{aligned}
\mathbf{K}_{wbwb}^{sL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,x}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{q=1}^{N_{wb}} EI \left(\frac{f\pi}{L} \right)^4 \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \frac{L}{2} \dot{w}_{fq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \frac{m\pi}{L} I_{pfm}^{scc} \right. \\
&\quad \left. + \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 \frac{m\pi}{L} I_{fpm}^{scc} \right] \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b e_c EA_s \left[\frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 \frac{m\pi}{L} I_{pfm}^{scc} \right. \\
&\quad \left. + \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 \frac{m\pi}{L} I_{fpm}^{scc} \right] \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \dot{w}_{pq}^b \quad (142) \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} w_{0m}^a w_{0r}^a EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{pfmr}^{cccc} \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0m}^a w_{0rs}^b 2EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} w_{0mn}^b w_{0rs}^b EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \dot{w}_{pq}^b
\end{aligned}$$

C.1.4 $\mathbf{K}_{uw}^{sNL,x}$ -matrix

$$\begin{aligned}
\mathbf{K}_{uawa}^{sNL,x} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sNL,x}}{\partial u_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \left(\frac{y_s}{b} \right)^3 I_{fpm}^{ccc} \right] \dot{w}_p^a \quad (143) \\
&\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{n\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_p^a
\end{aligned}$$

$$\begin{aligned}
\mathbf{K}_{uawb}^{sNL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,x}}{\partial u_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_{pq}^b
\end{aligned} \tag{144}$$

$$\begin{aligned}
\mathbf{K}_{ubwa}^{sNL,x} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sNL,x}}{\partial u_{fg}^b \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{g\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_p^a
\end{aligned} \tag{145}$$

$$\begin{aligned}
\mathbf{K}_{ubwb}^{sNL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,x}}{\partial u_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) I_{fpm}^{ccc} \right] \dot{w}_{pq}^b
\end{aligned} \tag{146}$$

$$\begin{aligned}
\mathbf{K}_{ucwa}^{sNL,x} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sNL,x}}{\partial u^c \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} w_p^a E A_s \left[\left(\frac{p\pi}{L} \right)^2 \left(\frac{y_s}{b} \right)^2 \frac{1}{2} \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wab}} \sum_{n=1}^{N_{wb}} w_{pn}^b E A_s \left[\left(\frac{p\pi}{L} \right)^2 \frac{y_s}{b} \sin\left(\frac{n\pi}{b} y_s\right) \frac{1}{2} \right] \dot{w}_p^a
\end{aligned} \tag{147}$$

$$\begin{aligned}
\mathbf{K}_{ucwb}^{sNL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,x}}{\partial u^c \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wab}} \sum_{q=1}^{N_{wb}} w_p^a E A_s \left[\left(\frac{p\pi}{L} \right)^2 \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \frac{1}{2} \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wab}} \sum_{q=1}^{N_{wb}} \sum_{n=1}^{N_{wb}} w_{pn}^b E A_s \left[\left(\frac{p\pi}{L} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \frac{1}{2} \right] \dot{w}_{pq}^b
\end{aligned} \tag{148}$$

C.1.5 $\mathbf{K}_{uww}^{sNL,x}$ -matrix

$$\begin{aligned}
\mathbf{K}_{uww}^{sNL,x} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sNL,x}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{i=1}^{M_{ua}} u_i^a E A_s \frac{i\pi}{L} \frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{y_s}{b}\right)^3 I_{ipf}^{ccc} \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{i=1}^{M_{ub}} \sum_{j=1}^{N_{ub}} u_{ij}^b E A_s \frac{i\pi}{L} \frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{y_s}{b}\right)^2 \sin\left(\frac{j\pi}{b} y_s\right) I_{ipf}^{ccc} \dot{w}_p^a \\
&\quad + u^c E A_s \frac{1}{2} \left(\frac{f\pi}{L}\right)^2 \left(\frac{y_s}{b}\right)^2 \dot{w}_f^a
\end{aligned} \tag{149}$$

$$\begin{aligned}
\mathbf{K}_{uww}^{sNL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,x}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{ua}} u_i^a E A_s \frac{i\pi}{L} \frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{y_s}{b}\right)^2 \sin\left(\frac{q\pi}{b} y_s\right) I_{ipf}^{ccc} \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{ub}} \sum_{j=1}^{N_{ub}} u_{ij}^b E A_s \frac{i\pi}{L} \frac{f\pi}{L} \frac{p\pi}{L} \frac{y_s}{b} \sin\left(\frac{j\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{ipf}^{ccc} \dot{w}_{pq}^b \\
&\quad + \sum_{q=1}^{N_{wb}} u^c E A_s \frac{1}{2} \left(\frac{f\pi}{L}\right)^2 \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \dot{w}_{fq}^b
\end{aligned} \tag{150}$$

$$\begin{aligned}
\mathbf{K}_{uww}^{sNL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,x}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{ua}} u_i^a E A_s \frac{i\pi}{L} \frac{f\pi}{L} \frac{p\pi}{L} \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{ipf}^{ccc} \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{ub}} \sum_{j=1}^{N_{ub}} u_{ij}^b E A_s \frac{i\pi}{L} \frac{f\pi}{L} \frac{p\pi}{L} \sin\left(\frac{j\pi}{b} y_s\right) \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{ipf}^{ccc} \dot{w}_{pq}^b \\
&\quad + \sum_{q=1}^{N_{wb}} u^c E A_s \frac{1}{2} \left(\frac{f\pi}{L}\right)^2 \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \dot{w}_{fq}^b
\end{aligned} \tag{151}$$

C.1.6 $\mathbf{K}_{ww}^{sNL,x}$ -matrix

$$\begin{aligned}
\mathbf{K}_{wawa}^{sNL,x} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sNL,x}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a e_c E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{m\pi}{L} \right)^2 I_{mfp}^{scc} \right. \\
&\quad \left. + \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 I_{fmp}^{scc} + \frac{m\pi}{L} \frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 I_{pfm}^{scc} \right] \left(\frac{y_s}{b} \right)^3 \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b e_c E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{m\pi}{L} \right)^2 I_{mfp}^{scc} \right. \\
&\quad \left. + \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 I_{fmp}^{scc} + \frac{m\pi}{L} \frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 I_{pfm}^{scc} \right] \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{n\pi}{b} y_s\right) \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} E A_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \left(\frac{y_s}{b} \right)^4 I_{pfmr}^{cccc} \left[3w_m^a w_{0r}^a + \frac{3}{2} w_m^a w_r^a \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} E A_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \quad \cdot \left(\frac{y_s}{b} \right)^3 \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \left[3w_m^a w_{0rs}^b + 3w_{0m}^a w_{rs}^b + 3w_m^a w_{rs}^b \right] \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} E A_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \quad \cdot \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{n\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \left[3w_{mn}^b w_{0rs}^b + \frac{3}{2} w_{mn}^b w_{rs}^b \right] \dot{w}_p^a
\end{aligned} \tag{152}$$

$$\begin{aligned}
\mathbf{K}_{wawb}^{sNL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,x}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a e_c E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{m\pi}{L} \right)^2 I_{mfp}^{scc} \right. \\
&\quad \left. + \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 I_{fmp}^{scc} + \frac{m\pi}{L} \frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 I_{pfm}^{scc} \right] \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) \dot{w}_{pq}^b \\
+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b e_c E A_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{m\pi}{L} \right)^2 I_{mfp}^{scc} \right. \\
&\quad \left. + \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 I_{fmp}^{scc} + \frac{m\pi}{L} \frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 I_{pfm}^{scc} \right] \frac{y_s}{b} \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \dot{w}_{pq}^b \\
+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} E A_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \left(\frac{y_s}{b} \right)^3 \sin\left(\frac{q\pi}{b} y_s\right) I_{pfmr}^{cccc} \left[3w_m^a w_{0r}^a + \frac{3}{2} w_m^a w_r^a \right] \dot{w}_{pq}^b \\
+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} E A_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \left[3w_m^a w_{0rs}^b + 3w_{0m}^a w_{rs}^b + 3w_m^a w_{rs}^b \right] \dot{w}_{pq}^b \\
+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} E A_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \frac{y_s}{b} \sin\left(\frac{n\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{pfmr}^{cccc} \left[3w_{mn}^b w_{0rs}^b + \frac{3}{2} w_{mn}^b w_{rs}^b \right] \dot{w}_{pq}^b
\end{aligned} \tag{153}$$

$$\begin{aligned}
\mathbf{K}_{wbwb}^{sNL,x} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,x}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b e_c EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{m\pi}{L} \right)^2 I_{mfp}^{scc} \right. \\
&\quad \left. + \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 I_{fmp}^{scc} + \frac{m\pi}{L} \frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 I_{pfm}^{scc} \right] \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a e_c EA_s \left[\frac{f\pi}{L} \frac{p\pi}{L} \left(\frac{m\pi}{L} \right)^2 I_{mfp}^{scc} \right. \\
&\quad \left. + \frac{m\pi}{L} \frac{p\pi}{L} \left(\frac{f\pi}{L} \right)^2 I_{fmp}^{scc} + \frac{m\pi}{L} \frac{f\pi}{L} \left(\frac{p\pi}{L} \right)^2 I_{pfm}^{scc} \right] \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \left(\frac{y_s}{b} \right)^2 \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{pfmr}^{cccc} \left[3w_m^a w_{0r}^a + \frac{3}{2} w_m^a w_r^a \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \frac{y_s}{b} \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) I_{pfmr}^{cccc} \left[3w_m^a w_{0rs}^b + 3w_{0m}^a w_{rs}^b + 3w_m^a w_{rs}^b \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} EA_s \frac{f\pi}{L} \frac{p\pi}{L} \frac{m\pi}{L} \frac{r\pi}{L} \\
&\quad \cdot \sin\left(\frac{g\pi}{b} y_s\right) \sin\left(\frac{n\pi}{b} y_s\right) \sin\left(\frac{s\pi}{b} y_s\right) \sin\left(\frac{q\pi}{b} y_s\right) I_{pfmr}^{cccc} \left[3w_{mn}^b w_{0rs}^b + \frac{3}{2} w_{mn}^b w_{rs}^b \right] \dot{w}_{pq}^b
\end{aligned} \tag{154}$$

C.2 $\mathbf{K}^{s,y}$ -matrices for stiffeners in y -direction

C.2.1 $\mathbf{K}_{vva}^{sL,y}$ -matrix

$$\begin{aligned}
\mathbf{K}_{vva}^{sL,y} \dot{\mathbf{v}}^a &= \frac{\partial^2 U^{sL,y}}{\partial v_f^a \partial v_p^a} \dot{v}_p^a \\
&= \sum_{p=1}^{M_{va}} EA_s \frac{1}{b} \cos\left(\frac{f\pi}{L} x_s\right) \cos\left(\frac{p\pi}{L} x_s\right) \dot{v}_p^a
\end{aligned} \tag{155}$$

$$\begin{aligned}
\mathbf{K}_{vbb}^{sL,y} \dot{\mathbf{v}}^b &= \frac{\partial^2 U^{sL,y}}{\partial v_{fg}^b \partial v_{pq}^b} \dot{v}_{pq}^b \\
&= \sum_{p=1}^{M_{vb}} EA_s \left(\frac{g\pi}{b} \right)^2 \frac{b}{2} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \dot{v}_{pg}^b
\end{aligned} \tag{156}$$

$$\mathbf{K}_{vcvc}^{sL,y} \dot{\mathbf{v}}^c = \frac{\partial^2 U^{sL,y}}{\partial v^c \partial v^c} \dot{v}^c = EA_s \frac{1}{b} \dot{v}^c \quad (157)$$

$$\mathbf{K}_{vavc}^{sL,y} \dot{\mathbf{v}}^c = \frac{\partial^2 U^{sL,y}}{\partial v_f^a \partial v^c} \dot{v}^c = EA_s \frac{1}{b} \cos\left(\frac{f\pi}{L} x_s\right) \dot{v}^c \quad (158)$$

$$\mathbf{K}_{vcva}^{sL,y} \dot{\mathbf{v}}^a = \frac{\partial^2 U^{sL,y}}{\partial v^c \partial v_p^a} \dot{v}_p^a = \sum_{p=1}^{M_{va}} EA_s \frac{1}{b} \cos\left(\frac{p\pi}{L} x_s\right) \dot{v}_p^a \quad (159)$$

$$\mathbf{K}_{vavb}^{sL,y} = 0, \quad \mathbf{K}_{vava}^{sL,y} = 0, \quad \mathbf{K}_{vavc}^{sL,y} = 0, \quad \mathbf{K}_{vavb}^{sL,y} = 0 \quad (160)$$

C.2.2 $\mathbf{K}_{vw}^{sL,y}$ -matrix

$$\mathbf{K}_{vawa}^{sL,y} \dot{\mathbf{w}}^a = \frac{\partial^2 U^{sL,y}}{\partial v_f^a \partial w_p^a} \dot{w}_p^a = \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0m}^a EA_s \left(\frac{1}{b}\right)^2 \cos\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_p^a \quad (161)$$

$$\begin{aligned} \mathbf{K}_{vawb}^{sL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,y}}{\partial v_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\ &= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} e_c EA_s \left(\frac{q\pi}{b}\right)^2 \frac{1}{b} I_q^s \cos\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \dot{w}_{pq}^b \\ &+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} w_{0mq}^b EA_s \left(\frac{q\pi}{b}\right)^2 \frac{1}{2} \cos\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \end{aligned} \quad (162)$$

$$\mathbf{K}_{vbwa}^{sL,y} \dot{\mathbf{w}}^a = \frac{\partial^2 U^{sL,y}}{\partial v_{fg}^b \partial w_p^a} \dot{w}_p^a = \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0mq}^b EA_s \left(\frac{g\pi}{b}\right)^2 \frac{1}{2} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_p^a \quad (163)$$

$$\begin{aligned} \mathbf{K}_{vbwb}^{sL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,y}}{\partial v_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\ &= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} e_c EA_s \left(\frac{q\pi}{b}\right)^2 \frac{g\pi}{b} I_{gq}^{cs} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \dot{w}_{pq}^b \\ &+ \sum_{p=1}^{M_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a EA_s \left(\frac{g\pi}{b}\right)^2 \frac{1}{2} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \\ &+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{M_{wb}} w_{0mn}^b EA_s \frac{g\pi}{b} \frac{q\pi}{b} \frac{n\pi}{b} I_{gqn}^{ccc} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \end{aligned} \quad (164)$$

$$\mathbf{K}_{vcwa}^{sL,y} \dot{\mathbf{w}}^a = \frac{\partial^2 U^{sL,y}}{\partial v^c \partial w_p^a} \dot{w}_p^a = \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_{0m}^a E A_s \left(\frac{1}{b}\right)^2 \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_p^a \quad (165)$$

$$\begin{aligned} \mathbf{K}_{vcwb}^{sL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,y}}{\partial v^c \partial w_{pq}^b} \dot{w}_{pq}^b \\ &= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} e_c E A_s \left(\frac{q\pi}{b}\right)^2 \frac{1}{b} I_q^s \sin\left(\frac{p\pi}{L} x_s\right) \dot{w}_{pq}^b \\ &+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} w_{0mq}^a E A_s \left(\frac{q\pi}{b}\right)^2 \frac{1}{2} \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \end{aligned} \quad (166)$$

C.2.3 $\mathbf{K}_{ww}^{sL,y}$ -matrix

$$\begin{aligned} \mathbf{K}_{wawa}^{sL,y} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sL,y}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\ &= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} w_{0m}^a w_{0r}^a E A_s \left(\frac{1}{b}\right)^3 J_{fpmr}^{ssss} \dot{w}_p^a \\ &+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} w_{0mn}^b w_{0rn}^b E A_s \frac{1}{b} \left(\frac{n\pi}{b}\right)^2 \frac{1}{2} J_{fpmr}^{ssss} \dot{w}_p^a \end{aligned} \quad (167)$$

$$\begin{aligned} \mathbf{K}_{wawb}^{sL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,y}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\ &= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a e_c E A_s \left[\left(\frac{1}{b}\right)^2 \left(\frac{q\pi}{b}\right)^2 I_q^s \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right)\right] \dot{w}_{pq}^b \\ &+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b e_c E A_s \left[\frac{1}{b} \left(\frac{q\pi}{b}\right)^2 \frac{n\pi}{b} I_{nq}^{cs} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right)\right] \dot{w}_{pq}^b \\ &+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wb}} w_{0m}^a w_{0rq}^b E A_s \frac{1}{b} \left(\frac{q\pi}{b}\right)^2 J_{fpmr}^{ssss} \dot{w}_{pq}^b \\ &+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} w_{0mn}^b w_{0rs}^b E A_s \frac{1}{b} \frac{q\pi}{b} \frac{n\pi}{b} \frac{s\pi}{b} J_{fpmr}^{ssss} I_{qns}^{ccc} \dot{w}_{pq}^b \end{aligned} \quad (168)$$

$$\begin{aligned}
\mathbf{K}_{wbwb}^{sL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sL,y}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \frac{EI}{2} \left(\frac{g\pi}{b} \right)^4 b \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \dot{w}_{pg}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_{0m}^a e_c EA_s \left[\frac{1}{b} \left(\frac{q\pi}{b} \right)^2 \frac{g\pi}{b} I_{qg}^{cs} \right. \\
&\quad \quad \quad \left. + \frac{1}{b} \left(\frac{g\pi}{b} \right)^2 \frac{q\pi}{b} I_{qg}^{cs} \right] \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{0mn}^b e_c EA_s \left[\left(\frac{q\pi}{b} \right)^2 \frac{g\pi}{b} \frac{n\pi}{b} I_{qgn}^{scc} \right. \\
&\quad \quad \quad \left. + \left(\frac{g\pi}{b} \right)^2 \frac{q\pi}{b} \frac{n\pi}{b} I_{qgn}^{scc} \right] \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} w_{0m}^a w_{0r}^a EA_s \frac{1}{b} \left(\frac{g\pi}{b} \right)^2 \frac{1}{2} J_{fpmr}^{ssss} \dot{w}_{pg}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{N_{wb}} \sum_{s=1}^{N_{wb}} w_{0m}^a w_{0rs}^b 2EA_s \frac{g\pi}{b} \frac{q\pi}{b} \frac{s\pi}{b} \frac{1}{b} J_{fpmr}^{ssss} I_{qgn}^{ccc} \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{N_{wb}} \sum_{s=1}^{N_{wb}} w_{0mn}^b w_{0rs}^b EA_s \frac{g\pi}{b} \frac{q\pi}{b} \frac{n\pi}{b} \frac{s\pi}{b} J_{fpmr}^{ssss} I_{qgn}^{cccc} \dot{w}_{pq}^b
\end{aligned} \tag{169}$$

C.2.4 $\mathbf{K}_{vw}^{sNL,y}$ -matrix

$$\mathbf{K}_{vawa}^{sNL,y} \dot{\mathbf{w}}^a = \frac{\partial^2 U^{sNL,y}}{\partial v_f^a \partial w_p^a} \dot{w}_p^a = \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a EA_s \left(\frac{1}{b} \right)^2 \cos\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_p^a \tag{170}$$

$$\mathbf{K}_{vawb}^{sNL,y} \dot{\mathbf{w}}^b = \frac{\partial^2 U^{sNL,y}}{\partial v_f^a \partial w_{pq}^b} \dot{w}_{pq}^b = \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} w_{mq}^b EA_s \left(\frac{q\pi}{b} \right)^2 \frac{1}{2} \cos\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \tag{171}$$

$$\mathbf{K}_{vbwa}^{sNL,y} \dot{\mathbf{w}}^a = \frac{\partial^2 U^{sNL,y}}{\partial v_{fg}^b \partial w_p^a} \dot{w}_p^a = \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} w_{mg}^b EA_s \left(\frac{g\pi}{b} \right)^2 \frac{1}{2} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_p^a \tag{172}$$

$$\begin{aligned}
\mathbf{K}_{vbw}^{sNL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,y}}{\partial v_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b E A_s \frac{g\pi}{b} \frac{q\pi}{b} \frac{n\pi}{b} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) I_{gqn}^{ccc} \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{m=1}^{M_{wa}} w_m^a E A_s \left(\frac{g\pi}{b}\right)^2 \frac{1}{2} \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pg}^b
\end{aligned} \tag{173}$$

$$\mathbf{K}_{vcwa}^{sNL,y} \dot{\mathbf{w}}^a = \frac{\partial^2 U^{sNL,y}}{\partial v^c \partial w_p^a} \dot{w}_p^a = \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} w_m^a E A_s \left(\frac{1}{b}\right)^2 \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_p^a \tag{174}$$

$$\mathbf{K}_{vcwb}^{sNL,y} \dot{\mathbf{w}}^b = \frac{\partial^2 U^{sNL,y}}{\partial v^c \partial w_{pq}^b} \dot{w}_{pq}^b = \sum_{p=1}^{M_{wb}} \sum_{m=1}^{M_{wb}} w_{mq}^b E A_s \left(\frac{g\pi}{b}\right)^2 \frac{1}{2} \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \tag{175}$$

C.2.5 $\mathbf{K}_{vw}^{sNL,y}$ -matrix

$$\begin{aligned}
\mathbf{K}_{vwawa}^{sNL,y} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sNL,y}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{i=1}^{M_{va}} v_i^a E A_s \left(\frac{1}{b}\right)^2 \cos\left(\frac{i\pi}{L} x_s\right) \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \dot{w}_p^a \\
&\quad + \sum_{p=1}^{M_{wb}} v^c E A_s \left(\frac{1}{b}\right)^2 \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{f\pi}{L} x_s\right) \dot{w}_p^a
\end{aligned} \tag{176}$$

$$\mathbf{K}_{vwa}^{sNL,y} \dot{\mathbf{w}}^b = \frac{\partial^2 U^{sNL,y}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b = \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{vbw}} \sum_{i=1}^{M_{vb}} v_{iq}^b E A_s \left(\frac{q\pi}{b}\right)^2 \frac{1}{2} \sin\left(\frac{i\pi}{L} x_s\right) \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \dot{w}_{pq}^b \tag{177}$$

$$\begin{aligned}
\mathbf{K}_{vbw}^{sNL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,y}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{i=1}^{M_{va}} v_i^a E A_s \left(\frac{g\pi}{b}\right)^2 \frac{1}{2} \cos\left(\frac{i\pi}{L} x_s\right) \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \dot{w}_{pg}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{i=1}^{M_{vb}} \sum_{j=1}^{N_{vb}} v_{ij}^b E A_s \frac{q\pi}{b} \frac{g\pi}{b} \frac{n\pi}{b} \sin\left(\frac{i\pi}{L} x_s\right) \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) I_{gqn}^{ccc} \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} v^c E A_s \left(\frac{g\pi}{b}\right)^2 \frac{1}{2} \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{f\pi}{L} x_s\right) \dot{w}_{pg}^b
\end{aligned} \tag{178}$$

C.2.6 $\mathbf{K}_{ww}^{sNL,y}$ -matrix

$$\begin{aligned}
\mathbf{K}_{wawa}^{sNL,y} \dot{\mathbf{w}}^a &= \frac{\partial^2 U^{sNL,y}}{\partial w_f^a \partial w_p^a} \dot{w}_p^a \\
&= \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} EA_s \left(\frac{1}{b}\right)^3 J_{fpmr}^{ssss} \left[3w_m^a w_{0r}^a + \frac{3}{2} w_m^a w_r^a \right] \dot{w}_p^a \\
&+ \sum_{p=1}^{M_{wa}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} EA_s \left(\frac{1}{b}\right)^2 \left(\frac{n\pi}{b}\right)^2 \frac{b}{2} J_{fpmr}^{ssss} \left[3w_{mn}^b w_{0rn}^b + \frac{3}{2} w_{mn}^b w_{rn}^b \right] \dot{w}_p^a
\end{aligned} \tag{179}$$

$$\begin{aligned}
\mathbf{K}_{wawb}^{sNL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,y}}{\partial w_f^a \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a e_c EA_s \left[\left(\frac{1}{b}\right)^2 \left(\frac{q\pi}{b}\right)^2 I_q^s \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b e_c EA_s \left[\frac{1}{b} \left(\frac{n\pi}{b}\right)^2 \frac{q\pi}{b} I_{qn}^{cs} \right. \\
&\quad \left. + \frac{1}{b} \left(\frac{q\pi}{b}\right)^2 \frac{n\pi}{b} I_{nq}^{cs} \right] \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wb}} EA_s \left(\frac{q\pi}{b}\right)^2 \left(\frac{1}{b}\right)^2 \frac{b}{2} J_{fmpqr}^{ssss} \left[3w_m^a w_{0rq}^b \right. \\
&\quad \left. + 3w_{0m}^a w_{rq}^b + 3w_m^a w_{rq}^b \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wb}} \sum_{s=1}^{N_{wb}} EA_s \frac{1}{b} \frac{q\pi}{b} \frac{n\pi}{b} \frac{s\pi}{b} J_{fpmr}^{ssss} \left[3w_{mn}^b w_{0rs}^b + \frac{3}{2} w_{mn}^b w_{rs}^b \right] \dot{w}_{pq}^b
\end{aligned} \tag{180}$$

$$\begin{aligned}
\mathbf{K}_{wbwb}^{sNL,y} \dot{\mathbf{w}}^b &= \frac{\partial^2 U^{sNL,y}}{\partial w_{fg}^b \partial w_{pq}^b} \dot{w}_{pq}^b \\
&= \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wa}} w_m^a e_c EA_s \left[\frac{1}{b} \left(\frac{q\pi}{b} \right)^2 \frac{g\pi}{b} I_{gq}^{cs} \right. \\
&\quad \left. + \frac{1}{b} \left(\frac{g\pi}{b} \right)^2 \frac{q\pi}{b} I_{gq}^{cs} \right] \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} w_{mn}^b e_c EA_s \left[\left(\frac{n\pi}{b} \right)^2 \frac{g\pi}{b} \frac{q\pi}{b} I_{ngq}^{scc} + \left(\frac{q\pi}{b} \right)^2 \frac{n\pi}{b} \frac{g\pi}{b} I_{qng}^{scc} \right. \\
&\quad \left. + \left(\frac{g\pi}{b} \right)^2 \frac{n\pi}{b} \frac{q\pi}{b} I_{gnq}^{scc} \right] \sin\left(\frac{f\pi}{L} x_s\right) \sin\left(\frac{p\pi}{L} x_s\right) \sin\left(\frac{m\pi}{L} x_s\right) \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{m=1}^{M_{wa}} \sum_{r=1}^{M_{wa}} EA_s \left(\frac{q\pi}{b} \right)^2 \left(\frac{1}{b} \right)^2 \frac{b}{2} J_{fpmr}^{ssss} \left[3w_{0m}^a w_r^a + \frac{3}{2} w_m^a w_r^a \right] \dot{w}_{pq}^b \\
&\quad + \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} EA_s \frac{1}{b} \frac{q\pi}{b} \frac{g\pi}{b} \frac{n\pi}{b} J_{fpmr}^{ssss} I_{gqn}^{ccc} \left[3w_r^a w_{0mn}^b + 3w_{0r}^a w_{mn}^b + 3w_r^a w_{mn}^b \right] \dot{w}_{pq}^b \\
&+ \sum_{p=1}^{M_{wb}} \sum_{q=1}^{N_{wb}} \sum_{m=1}^{M_{wb}} \sum_{n=1}^{N_{wb}} \sum_{r=1}^{M_{wa}} \sum_{s=1}^{N_{wb}} EA_s \frac{g\pi}{b} \frac{q\pi}{b} \frac{n\pi}{b} \frac{s\pi}{b} J_{fpmr}^{ssss} I_{gqns}^{cccc} \left[3w_{mn}^b w_{0rs}^b + \frac{3}{2} w_{mn}^b w_{rs}^b \right] \dot{w}_{pq}^b
\end{aligned} \tag{181}$$

D Integrals and expressions

$$I_j^s = \int_0^b \sin\left(\frac{j\pi}{b} y\right) dy = \frac{b}{j\pi} [1 - (-1)^j] \tag{182}$$

$$I_j^{ys} = \int_0^b \frac{y}{b} \sin\left(\frac{j\pi}{b} y\right) dy = -\frac{b}{j\pi} (-1)^j \tag{183}$$

$$I_j^{yc} = \int_0^b \frac{y}{b} \cos\left(\frac{j\pi}{b} y\right) dy = \frac{b}{(j\pi)^2} [(-1)^j - 1] \tag{184}$$

$$I_j^{yys} = \int_0^b \left(\frac{y}{b}\right)^2 \sin\left(\frac{j\pi}{b} y\right) dy = \frac{b}{(j\pi)^3} \left([2 - (j\pi)^2] (-1)^j - 2 \right) \tag{185}$$

$$I_j^{yyc} = \int_0^b \left(\frac{y}{b}\right)^2 \cos\left(\frac{j\pi}{b} y\right) dy = \frac{2b}{(j\pi)^2} (-1)^j \tag{186}$$

$$I_j^{yyyy} = \int_0^b \left(\frac{y}{b}\right)^3 \sin\left(\frac{j\pi}{b}y\right) dy = \frac{b}{(j\pi)^3} [6 - (j\pi)^2] (-1)^j \quad (187)$$

$$I_{jl}^{cs} = \int_0^b \cos\left(\frac{j\pi}{b}y\right) \sin\left(\frac{l\pi}{b}y\right) dy = \begin{cases} 0 & \text{if } j = l \\ \frac{b[1 - (-1)^j (-1)^l] l}{\pi(l^2 - j^2)} & \text{if } j \neq l \end{cases} \quad (188)$$

$$I_{jl}^{ycc} = \int_0^b \frac{y}{b} \cos\left(\frac{j\pi}{b}y\right) \cos\left(\frac{l\pi}{b}y\right) dy = \begin{cases} \frac{b}{4} & \text{if } j = l \\ \frac{b[j^2 + l^2][(-1)^{j+l} - 1]}{\pi^2(j^2 - l^2)^2} & \text{if } j \neq l \end{cases} \quad (189)$$

$$I_{jl}^{yss} = \int_0^b \frac{y}{b} \sin\left(\frac{j\pi}{b}y\right) \sin\left(\frac{l\pi}{b}y\right) dy = \begin{cases} \frac{b}{4} & \text{if } j = l \\ \frac{b2jl}{\pi^2(j^2 - l^2)^2} [(-1)^{j+l} - 1] & \text{if } j \neq l \end{cases} \quad (190)$$

$$I_{jl}^{yccs} = \int_0^b \frac{y}{b} \cos\left(\frac{j\pi}{b}y\right) \sin\left(\frac{l\pi}{b}y\right) dy = \begin{cases} -\frac{b}{4\pi j} & \text{if } j = l \\ \frac{b(-1)^j (-1)^l [j^2 l - l^3]}{\pi(j^2 - l^2)^2} & \text{if } j \neq l \end{cases} \quad (191)$$

$$I_{jl}^{yyss} = \int_0^b \left(\frac{y}{b}\right)^2 \sin\left(\frac{j\pi}{b}y\right) \sin\left(\frac{l\pi}{b}y\right) dy = \begin{cases} \frac{b[2(\pi j)^2 - 3]}{12(\pi j)^2} & \text{if } j = l \\ \frac{b4[j^3 l - l^3 j](-1)^{j+l}}{\pi^2(j^6 - l^6 + 3j^2 l^4 - 3j^4 l^2)} & \text{if } j \neq l \end{cases} \quad (192)$$

$$I_{jl}^{yycc} = \int_0^b \left(\frac{y}{b}\right)^2 \cos\left(\frac{j\pi}{b}y\right) \cos\left(\frac{l\pi}{b}y\right) dy = \begin{cases} \frac{b[3 + 2(\pi j)^2]}{12(\pi j)^2} & \text{if } j = l \\ \frac{b2[j^4 - l^4](-1)^{j+l}}{\pi^2(j^6 - l^6 + 3j^2 l^4 - 3j^4 l^2)} & \text{if } j \neq l \end{cases} \quad (193)$$

$$I_{jln}^{sss} = \int_0^b \sin\left(\frac{j\pi}{b}y\right) \sin\left(\frac{l\pi}{b}y\right) \sin\left(\frac{n\pi}{b}y\right) dy = \begin{cases} 0 & \text{if } \frac{j^4 + l^4 + n^4}{2(j^2 l^2 + j^2 n^2 + l^2 n^2)} = 1 \\ \frac{b2jln[(-1)^j (-1)^l (-1)^n - 1]}{\pi[j^4 + l^4 + n^4 - 2(j^2 l^2 + j^2 n^2 + l^2 n^2)]} & \text{otherwise} \end{cases} \quad (194)$$

$$I_{jln}^{ccc} = \int_0^b \cos\left(\frac{j\pi}{b}y\right) \cos\left(\frac{l\pi}{b}y\right) \cos\left(\frac{n\pi}{b}y\right) dy = \begin{cases} \frac{b}{4} & \text{if } \frac{j^4 + l^4 + n^4}{2(j^2 l^2 + j^2 n^2 + l^2 n^2)} = 1 \\ 0 & \text{otherwise} \end{cases} \quad (195)$$

$$I_{jln}^{css} = \int_0^b \cos\left(\frac{j\pi}{b}y\right) \sin\left(\frac{l\pi}{b}y\right) \sin\left(\frac{n\pi}{b}y\right) dy = \begin{cases} \frac{b}{4} & \text{if } \frac{j^4 + l^4 + n^4}{2(j^2 l^2 + j^2 n^2 + l^2 n^2)} = 1 \text{ and } l^2 + n^2 > j^2 \\ -\frac{b}{4} & \text{if } \frac{j^4 + l^4 + n^4}{2(j^2 l^2 + j^2 n^2 + l^2 n^2)} = 1 \text{ and } l^2 + n^2 < j^2 \\ 0 & \text{otherwise} \end{cases} \quad (196)$$

$$I_{jln}^{scc} = \int_0^b \sin\left(\frac{j\pi}{b}y\right)\cos\left(\frac{l\pi}{b}y\right)\cos\left(\frac{n\pi}{b}y\right) dy = \begin{cases} 0 & \text{if } \frac{j^4+l^4+n^4}{2(j^2l^2+j^2n^2+l^2n^2)} = 1 \\ \frac{bj[l^2+n^2-j^2][(-1)^j(-1)^l(-1)^{n-1}]}{\pi[j^4+l^4+n^4-2(j^2l^2+j^2n^2+l^2n^2)]} & \text{otherwise} \end{cases} \quad (197)$$

$$I_{jln}^{ysss} = \int_0^b \frac{y}{b} \sin\left(\frac{j\pi}{b}y\right)\sin\left(\frac{l\pi}{b}y\right)\sin\left(\frac{n\pi}{b}y\right) dy = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4 \quad (198)$$

where

$$\Phi_1 = \begin{cases} 0 & \text{if } n+l-j=0 \\ -\frac{b(-1)^{j+l-j}}{4\pi(n+l-j)} & \text{otherwise} \end{cases}, \quad \Phi_2 = \begin{cases} 0 & \text{if } n+j-l=0 \\ -\frac{b(-1)^{n+j-l}}{4\pi(n+j-l)} & \text{otherwise} \end{cases} \quad (199)$$

$$\Phi_3 = \begin{cases} 0 & \text{if } n-j-l=0 \\ \frac{b(-1)^{n-j-l}}{4\pi(n-j-l)} & \text{otherwise} \end{cases}, \quad \Phi_4 = \frac{b(-1)^{n+j+l}}{4\pi(n+j+l)} \quad (200)$$

$$I_{jln}^{yscc} = \int_0^b \frac{y}{b} \sin\left(\frac{j\pi}{b}y\right)\cos\left(\frac{l\pi}{b}y\right)\cos\left(\frac{n\pi}{b}y\right) dy = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 \quad (201)$$

where

$$\Psi_1 = \begin{cases} 0 & \text{if } j-l-n=0 \\ -\frac{b(-1)^{j-l-n}}{4\pi(j-l-n)} & \text{otherwise} \end{cases}, \quad \Psi_2 = \begin{cases} 0 & \text{if } j-l+n=0 \\ -\frac{b(-1)^{j-l+n}}{4\pi(j-l+n)} & \text{otherwise} \end{cases} \quad (202)$$

$$\Psi_3 = \begin{cases} 0 & \text{if } j+l-n=0 \\ -\frac{b(-1)^{j+l-n}}{4\pi(j+l-n)} & \text{otherwise} \end{cases}, \quad \Psi_4 = -\frac{b(-1)^{n+j+l}}{4\pi(n+j+l)} \quad (203)$$

$$I_{jlns}^{cccc} = \int_0^b \cos\left(\frac{j\pi}{b}y\right)\cos\left(\frac{l\pi}{b}y\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{s\pi}{b}y\right) dy = \Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 + \Theta_6 + \Theta_7 \quad (204)$$

where

$$\Theta_1 = \begin{cases} \frac{b}{8} & \text{if } j+l=n+s \\ 0 & \text{otherwise} \end{cases}, \quad \Theta_2 = \begin{cases} \frac{b}{8} & \text{if } j+l=-n+s \\ 0 & \text{otherwise} \end{cases} \quad (205)$$

$$\Theta_3 = \begin{cases} \frac{b}{8} & \text{if } j+l=n-s \\ 0 & \text{otherwise} \end{cases}, \quad \Theta_4 = \begin{cases} \frac{b}{8} & \text{if } j-l=-n-s \\ 0 & \text{otherwise} \end{cases} \quad (206)$$

$$\Theta_5 = \begin{cases} \frac{b}{8} & \text{if } j-l=n+s \\ 0 & \text{otherwise} \end{cases}, \quad \Theta_6 = \begin{cases} \frac{b}{8} & \text{if } j-l=-n+s \\ 0 & \text{otherwise} \end{cases} \quad (207)$$

$$\Theta_7 = \begin{cases} \frac{b}{8} & \text{if } j - l = n - s \\ 0 & \text{otherwise} \end{cases} \quad (208)$$

$$I_{jlns}^{ssss} = \int_0^b \sin\left(\frac{j\pi}{b}y\right)\sin\left(\frac{l\pi}{b}y\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{s\pi}{b}y\right) dy = \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_7 \quad (209)$$

where

$$\Omega_1 = \begin{cases} -\frac{b}{8} & \text{if } j + l = -n + s \\ 0 & \text{otherwise} \end{cases}, \quad \Omega_2 = \begin{cases} -\frac{b}{8} & \text{if } j + l = n - s \\ 0 & \text{otherwise} \end{cases} \quad (210)$$

$$\Omega_3 = \begin{cases} \frac{b}{8} & \text{if } j + l = n + s \\ 0 & \text{otherwise} \end{cases}, \quad \Omega_4 = \begin{cases} \frac{b}{8} & \text{if } j - l = -n + s \\ 0 & \text{otherwise} \end{cases} \quad (211)$$

$$\Omega_5 = \begin{cases} \frac{b}{8} & \text{if } j - l = n - s \\ 0 & \text{otherwise} \end{cases}, \quad \Theta_6 = \begin{cases} -\frac{b}{8} & \text{if } j - l = -n - s \\ 0 & \text{otherwise} \end{cases} \quad (212)$$

$$\Omega_7 = \begin{cases} -\frac{b}{8} & \text{if } j - l = n + s \\ 0 & \text{otherwise} \end{cases} \quad (213)$$

$$I_{jlns}^{sscc} = \int_0^b \sin\left(\frac{j\pi}{b}y\right)\sin\left(\frac{l\pi}{b}y\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{s\pi}{b}y\right) dy = \Upsilon_1 + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_5 + \Upsilon_6 + \Upsilon_7 \quad (214)$$

where

$$\Upsilon_1 = \begin{cases} -\frac{b}{8} & \text{if } j + l = n + s \\ 0 & \text{otherwise} \end{cases}, \quad \Upsilon_2 = \begin{cases} -\frac{b}{8} & \text{if } j + l = -n + s \\ 0 & \text{otherwise} \end{cases} \quad (215)$$

$$\Upsilon_3 = \begin{cases} -\frac{b}{8} & \text{if } j + l = n - s \\ 0 & \text{otherwise} \end{cases}, \quad \Upsilon_4 = \begin{cases} \frac{b}{8} & \text{if } j - l = -n - s \\ 0 & \text{otherwise} \end{cases} \quad (216)$$

$$\Upsilon_5 = \begin{cases} \frac{b}{8} & \text{if } j - l = n + s \\ 0 & \text{otherwise} \end{cases}, \quad \Upsilon_6 = \begin{cases} \frac{b}{8} & \text{if } j - l = -n + s \\ 0 & \text{otherwise} \end{cases} \quad (217)$$

$$\Upsilon_7 = \begin{cases} \frac{b}{8} & \text{if } j - l = n - s \\ 0 & \text{otherwise} \end{cases} \quad (218)$$

$$J_{ikmr}^{ssss} = \sin\left(\frac{i\pi}{L}x_s\right)\sin\left(\frac{k\pi}{L}x_s\right)\sin\left(\frac{m\pi}{L}x_s\right)\sin\left(\frac{r\pi}{L}x_s\right) \quad (219)$$